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HYDRODYNAMIC DESCRIPTION OF A STRONGLY-RAREFIED PLASMA

T. F. Volkov

1. A strict description of the behavior of a plasma must be carried out with the aid of the kinetic equations for electrons and ions. This method, however, is very complex, and in many cases which are of practical importance it is not necessary. As can be shown by theory (Ref. 1), a system of kinetic equations can be replaced by a simpler system of so-called transfer equations for local, macroscopic quantities determining the behavior of electrons and ions, if two basic conditions are fulfilled: /3*

1) Many collisions occur during the characteristic time of the process;
2) The course traversed by the particles between two collisions is considerably less than the distance over which the macroscopic quantities change substantially.

Frequently these conditions are not fulfilled. Thus, for a typical "thermonuclear" plasma, for example, the second condition is violated. An approximate description of the plasma behavior can then be obtained if, in general, the collision terms are disregarded in the kinetic equations, i.e., if we start with an equation of the following type:

$$\frac{\partial f}{\partial t} + (\mathbf{v}\nabla) f + \frac{e}{m}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right\}\frac{\partial f}{\partial \mathbf{v}} = 0 \qquad (1)$$

(actually, two equations for electrons and ions must be examined, together with the Maxwell equations). The equations of hydrodynamics, however, are simpler and make it possible to interpret the solutions obtained more clearly. It will be shown below that a hydrodynamic description of a plasma without collisions can in several cases be used at least as a heuristic method for obtaining results which are qualitatively correct.

Let us try to obtain the equation of hydrodynamics from the kinetic equation (1) by the usual method, taking the moments of the distribution function f. For this purpose, it is convenient to factor out the mean velocity $\mathbf{V} = \int \mathbf{v} f \, d\mathbf{v}$, changing to the new variable

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*Note: Numbers in the margin indicate pagination in the original foreign text.

$\mathbf{w} = \mathbf{v} - \mathbf{V}$. It is convenient to carry out additional calculations in the coordinate form. The change to the new variables in velocity space is reduced to the substitution /4

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \frac{\partial V_k}{\partial t}\frac{\partial}{\partial W_k}, \quad \frac{\partial}{\partial x_j} \to \frac{\partial}{\partial x_j} - \frac{\partial V_k}{\partial x_j}\cdot\frac{\partial}{\partial w_k}, \quad \frac{\partial}{\partial v_j} \to \frac{\partial}{\partial w_j}.$$

Also utilizing the designations

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V_j\frac{\partial}{\partial x_j} \text{ и } F_j = \frac{e}{m}\left(E_j + \varepsilon_{jkl}\frac{V_k}{c}B_l\right),$$

(ε_{jkl} is the completely antisymmetric tensor of the third rank), we reduce the kinetic equation to the form

$$\frac{df}{dt} + w_j\frac{\partial f}{\partial x_j} + \left(F_k - \frac{dV_k}{dt}\right)\frac{\partial f}{\partial w_k} - \frac{\partial V_k}{\partial x_j}w_j\frac{\partial f}{\partial w_k} + \\ + \varepsilon_{jkl}\frac{w_k}{c}B_l\frac{\partial f}{\partial w_j} = 0. \tag{2}$$

Multiplying this equation by 1 and w_i, and integrating with respect to $\mathbf{w}$, we obtain the equation of continuity and the equation of motion

$$\frac{dn}{dt} + n\frac{\partial V_k}{\partial x_k} = 0, \tag{3}$$

$$\frac{dV_i}{dt} = -\frac{\partial p_{ij}}{\partial x_j} + F_i \tag{4}$$

or in vector form

$$\frac{dn}{dt} + n\,\mathrm{div}\,\mathbf{V} = 0,$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\,\mathbf{V} = -\mathrm{div}\,\mathbf{P} + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{VB}]\right).$$

Here $n = \int f\,d\mathbf{w}$; $\mathbf{P}$ is the pressure tensor with the components $p_{ij} = m\int w_i w_j f\,d\mathbf{w}$. In order to express the quantities p_{ij} in terms of hydrodynamic variables (density, velocity), it is possible to try and obtain the equation for the second moments of the distribution function. Multiplying equation (2) by $w_m w_n$ and integrating with respect to velocity, we find

$$\frac{dp_{mn}}{dt} + \frac{\partial}{\partial x_j}p_{mnj} + p_{mn}\frac{\partial V_k}{\partial x_k} + \frac{\partial V_m}{\partial x_j}p_{nj} + \frac{\partial V_n}{\partial x_j}p_{mj} - \\ - \varepsilon_{mkl}p_{nk}B_l - \varepsilon_{nkl}p_{mk}B_l. \tag{5}$$

This equation can be interpreted as an equation of state. We did not obtain a closed system of equations, because the third moments $p_{mnj} = \int w_m w_n w_j f \, dw$ entered into equation (5), and the fourth moments enter into the equation for the third moments, etc. In order to obtain a closed system of equations, it is necessary to know the distribution function f, which is not determined in this scheme. We should note here that in the opposite case, when the particle collisions play a definite role, the distribution function is single-valued (Ref. 2). In order to eliminate the third moments, assumptions are usually introduced concerning the symmetry of the distribution function. For a sufficiently isotropic distribution function (for example, a velocity distribution which is close to the Maxwellian one), the third moments can be small or even exactly equal to zero. The non-diagonal components of the tensor p_{ij} are small also, i.e. /5

$$p_{ij} = p\delta_{ij}. \tag{6}$$

Substituting expression (6) in equation (5) we obtain the equation of state

$$\frac{dp}{dt} + \frac{5}{3} p \operatorname{div} \mathbf{V} = 0. \tag{7}$$

Expressing div V from the equation of continuity, we find

$$\frac{d}{dt} \frac{p}{n^{5/3}} = 0, \tag{8}$$

i.e., the customary adiabatic law with $\gamma = \frac{5}{3}$ holds.

Some times distinct anisotropy exists in the distribution of the particles - for example, in the presence of a strong magnetic field (see below). Since an exchange of energy between degrees of freedom does not occur in a plasma with infrequent collisions, the pressure along the magnetic field can differ from the pressure in the perpendicular direction ($p_{\parallel} \neq p_{\perp}$). In a system of Cartesian coordinates, one of the axes of which is directed along the magnetic field, the pressure tensor has a diagonal form

$$(p_{mn}) = \begin{pmatrix} p_{\parallel} & 0 & 0 \\ 0 & p_{\perp} & 0 \\ 0 & 0 & p_{\perp} \end{pmatrix}.$$

In an arbitrary system of coordinates, its components will be $p_{mn} = p_{\parallel} \tau_m \tau_n + p_{\perp} (\delta_{mn} - \tau_m \tau_n)$, where τ_i are the components of the unit vector directed along the magnetic field. Substituting p_{mn}

in equation (5), for $p\|$ and $p\perp$ we obtain different equations of state (assuming the third moments to be small)

$$\frac{dp_{\|}}{dt} + p_{\|} \operatorname{div} \mathbf{V} + 2p_{\|} \tau (\tau \nabla) \mathbf{V} = 0; \tag{9}$$

$$\frac{dp_{\perp}}{dt} + 2p_{\perp} \operatorname{div} \mathbf{V} - p_{\perp} \tau (\tau \nabla) \mathbf{V} = 0. \tag{10}$$

An interpretation of these equations will be given below.

2. The hydrodynamic equations which are obtained have a formal meaning, because nowhere in their derivation is there proof that a plasma without collisions can be regarded as a continuous medium. /6
In terms of absolute completeness, this statement is simply not true. Thus, for example, the pressure tensor which appears in the hydrodynamic equations (4) and (5) is the convective transfer of impulse by the particles, and not the force which one particle of the plasma exerts upon another. Similarly, the velocity V is the mean velocity of 1 cm^3 of a collection of non-interacting particles, and not the velocity of an elementary volume of substance in ordinary hydrodynamics. The application of the system of hydrodynamic equations (3) - (5) to a neutral gas without collisions has no meaning. This does have meaning for a plasma, because *the self-consistent fields* play an essential role in the plasma processes. These fields replace the collisions, connecting the particles, thus making it difficult to separate the individual particles.

Let us show, by way of an example, that in certain cases which are important in a practical sense, the velocities of electrons differ very little from the velocities of the ions, primarily due to the presence of a self-consistent magnetic field (Ref. 11) (the electrons "are bound" to the ions even in the absence of collisions). For this relationship we can have the approximation

$$\frac{|V_i - V_e|}{V_i} = \frac{j}{enV_i}.$$

We should note that from the Maxwell equations $j \sim \frac{cB}{4\pi L}$ (L is the characteristic dimension of the field inhomogeneity). If the relationship of the kinetic energy density of the particles to the magnetic energy density is designated by $\alpha \equiv \frac{4\pi m_i n V_i^2}{B^2}$, then

$$\frac{|V_i - V_e|}{V_i} \approx \left(\frac{c^2 m_i}{4\pi e^2 L n}\right)^{1/2} = \frac{1}{\sqrt{\alpha \Pi}}.$$

The number Π is the so-called "linear proton number", i.e., the number of ions in a layer with a height equal to the classical radius

of the ion, and with a surface equal to the square of the characteristic dimension of the system. The relationship $|V_i - V_e|/V_i$ is small if $\sqrt{\alpha \Pi} >> 1$. In astrophysical applications, $L \sim 10^{10} - 10^{20}$, $\alpha \sim 1$, $\Pi >> 1$. For an experimental plasma, for $L \sim 10 \div 100$ cm, Π can be on the order of 1, and the assumption that the electrons are bound is not always true.

We should note that the hydrodynamic equations very roughly take into account the thermal motion of the particles. A certain average effect of the thermal motion is transmitted by the terms containing pressure. The actual role of thermal motion can only be established from the kinetic solution of the problem. Unfortunately, it is impossible to indicate universal criteria for the applicability of a hydrodynamic solution of the problemsof plasma dynamics. For an important special case of small fluctuations, for example, hydrodynamics provide at least qualitatively correct results, if the phase velocity of the waves is greater than the thermal velocity of the particles, i.e., small dissipative processes [Landau damping (Ref. 3)]. This is realized, for example, for Langmuir fluctuations or for ionic sound for /7
$T_e >> T_i$ (in the latter case, hydrodynamic examination yields the phase velocity $v_\Phi = \left(\frac{T_e + T_i}{m_i}\right)^{1/2}$ which is considerably larger than the thermal velocity of the ions for $T_e >> T_i$. An analogous situation occurs for magneto-sonic fluctuations.) The small value of the Landau damping (an effect which is determined completely by the thermal dispersion of velocities) can be explained by the fact that for $v_\Phi >> v_T$ the so-called resonance particles - which move together with the waves and are capable of strong interaction with them - are small.

3. Let us now examine the hydrodynamic equations which describe the behavior of a rarefied plasma in a strong magnetic field. We shall assume that the plasma is "magnetized", i.e., the Larmor radius of the particles is much less than the characteristic length of the plasma inhomogeneity, and the corresponding Larmor particles are larger than the characteristic particles of the process. In a strong magnetic field the distribution of the electrons and ions can have axial symmetry with respect to the direction of the latter. This circumstance leads to the fact that, if thermal currents are not present along the lines of force, the slow movements of the plasma conform to the equations of magnetic hydrodynamics with a non-isotropic pressure tensor. We shall give these equations below, starting with the kinetic equations for electrons and ions.

A kinetic equation in the absence of collision terms represents an equation of continuity in phase space $(\mathbf{r}, \mathbf{v}, t)$

$$\frac{\partial f}{\partial t} + \operatorname{div}_{\mathbf{r}} \mathbf{v} f + \operatorname{div}_{\mathbf{v}} \mathbf{a} f = 0. \qquad (11)$$

The symbols div_r and div_v designate the taking of divergence in spaces of coordinates and velocities, respectively; a represents the acceleration of the particle, f is the distribution function. For purposes of definiteness, let us examine the kinetic equation for a single type of particle. The results which are obtained can be used both for ions and for electrons with the aid of the substitution $e \to e_\alpha$ and $m \to m_\alpha$ ($\alpha = e, i$ for electrons and ions respectively).

We can write the kinetic equation (11) in the cylindrical system of coordinates in velocity space

$$\frac{\partial f}{\partial t} + \mathrm{div}\, \mathbf{v} f + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp a_\perp f + \frac{\partial}{\partial v_\parallel} a_\parallel f + \frac{1}{v_\perp}\frac{\partial}{\partial \Theta} a_\Theta f = 0. \qquad (12)$$

here $v_{||}$ is the component of velocity parallel to the magnetic field; $v_\perp$ is the component of velocity perpendicular to the magnetic field.

In addition, let us examine the equation of motion for a particle

$$\frac{d\mathbf{v}}{dt} = \mathbf{F} + \omega_B [\mathbf{v}\tau]. \qquad (13)$$

here /8

$$\omega_B = \frac{e\tilde{B}}{mc}, \quad \mathbf{F} = \frac{e\mathbf{E}}{m}, \quad \tau = \frac{\mathbf{B}}{B}.$$

the velocity $\mathbf{v}$ can be written in the form

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{v}_F + v_\parallel \tau + v_\perp (\tau_1 \cos\Theta + \tau_2 \sin\Theta). \qquad (14)$$

Here τ_1 and τ_2 are unit vectors which form a right-handed triplet and which are perpendicular to τ and to each other. The direction of these vectors does not matter to us, since they do not enter into the final results. The quantity $\mathbf{v}_F$ represents the velocity of electric drift:

$$\mathbf{v}_F = \frac{1}{\omega_B}[\mathbf{F}\tau] = \frac{c[\mathbf{EB}]}{B^2}. \qquad (15)$$

This is the velocity with which the charged particles drift in crossed electric and magnetic fields; $\mathbf{v}_F$ is also the velocity of the coordinate system in which the component of the electric field ($\mathbf{E}||\mathbf{B}$) which is perpendicular to τ, is not present. In the theory of drift motion of particles in a strong electric field (Ref. 4), the velocity $\mathbf{v}_F$ is assumed to factor out, by introducing a moving local coordinate system according to formula (14).

Substituting expression (14) in equation (13), we obtain

$$\dot{v}_\parallel \tau + v_\parallel \dot{\tau} + \mathbf{v}_F + \dot{v}_\perp (\tau_1 \cos\Theta + \tau_2 \sin\Theta) + v_\perp (\dot{\tau}_1 \cos\Theta + \dot{\tau}_2 \sin\Theta) +$$
$$+ v_\perp \dot{\Theta} (-\tau_1 \sin\Theta + \tau_2 \cos\Theta) = -\omega_B v_\perp (-\tau_1 \sin\Theta + \tau_2 \cos\Theta) + \tau (\mathbf{F}\tau).$$

Projecting in the directions $\tau, \tau_1 \cos\theta + \tau_2 \sin\theta$ and $-\tau_1 \sin\theta + \tau_2 \cos\theta$, we obtain

$$a_{\parallel} = \dot{v}_{\parallel} = (\tau,\ \mathrm{F} - \mathrm{v}_F') + \frac{v_{\perp}^2}{2}\operatorname{div}\tau - v_{\perp}\left[(\tau\tau_1') + \tau(\tau_1\nabla)\,\mathrm{v}_F\right]\cos\theta - \\ - v_{\perp}\left[(\tau\tau_2') + \tau(\tau_2\nabla)\,\mathrm{v}_F\right]\sin\theta + \ldots, \tag{16a}$$

$$a_{\perp} = \dot{v}_{\perp} = -\frac{v_{\perp}v_{\parallel}}{2}\operatorname{div}\tau - \frac{v_{\perp}}{2}\operatorname{div}\mathrm{v}_F + \frac{v_{\perp}}{2}\tau(\tau\nabla)\,\mathrm{v}_F - \\ - \tau_1(v_{\parallel}\tau' + \mathrm{v}_F')\cos\theta - \tau_2(v_{\parallel}\tau' + \mathrm{v}_F')\sin\theta + \ldots, \tag{16b}$$

$$a_{\theta} = \dot{\theta} = -\omega_B + \frac{\tau_2}{v_{\perp}}\cos\theta\left\{-v_{\parallel}\tau' - \mathrm{v}_F' - v_{\perp}^{\cdot}(\tau_1\nabla)\,\tau_1\right\} + \\ + \frac{\tau_1}{v_{\perp}}\sin\theta\left\{v_{\parallel}\tau' + \mathrm{v}_F' + v_{\perp}^2(\tau_2\nabla)\,\tau_2\right\} + \ldots \tag{16c}$$

[the derivative $\frac{\partial}{\partial t} + (\mathrm{v}_F + v_{\parallel}\tau,\ \nabla)$ is designated by the prime sign]. In formulas (16a, b, c) the terms which are proportional to $\cos 2\theta$ and $\sin 2\theta$, are not shown, and in equation (16c) neither is the term shown which is independent of θ and does not contain ω_B. Their explicit form is not essential for future purposes.

Let us substitute the expressions for $\alpha_{\parallel}$, $\alpha_{\perp}$ and α_{θ} from equations (16a, b, c) into the kinetic equation (12); let us reduce it to the form /9

$$\left\{D_0 + A_0\frac{\partial}{\partial\theta} + \cos\theta\left(D_1 + A_1\frac{\partial}{\partial\theta}\right) + \sin\theta\left(D_2 + A_2\frac{\partial}{\partial\theta}\right) + \right. \\ \left. + \cos 2\theta\left(D_3 + A_3\frac{\partial}{\partial\theta}\right) + \sin 2\theta\left(D_4 + A_4\frac{\partial}{\partial\theta}\right)\right\} f = \omega_B\frac{\partial f}{\partial\theta}. \tag{17}$$

here

$$D_0 = \frac{\partial}{\partial t} + \operatorname{div}(\mathrm{v}_F + v_{\parallel}\tau) + \frac{\partial}{\partial v_{\parallel}}\left\{(\tau_1\mathrm{F} - \mathrm{v}_F') + \frac{v_{\perp}^2}{2}\operatorname{div}\tau\right\} + \\ + \frac{1}{v_{\perp}}\frac{\partial}{\partial v_{\perp}}v_{\perp}^2\left\{-\frac{v_{\parallel}}{2}\operatorname{div}\tau - \frac{1}{2}\operatorname{div}\tau + \frac{1}{2}\tau(\tau\nabla)\,v_F\right\}; \tag{17a}$$

$$D_1 \equiv v_{\perp}\operatorname{div}\tau_1 - \frac{\partial}{\partial v_{\parallel}}v_{\perp}\left\{(\tau\tau_1') + \tau(\tau_1\nabla)\,\mathrm{v}_F\right\} - \\ - \frac{1}{v_{\perp}}\frac{\partial}{\partial v_{\perp}}v_{\perp}\tau_1(v_{\parallel}\tau' + \mathrm{v}_F') + \frac{\tau_1'}{v_{\perp}}\left\{v_{\parallel}\tau' + \mathrm{v}_F' + v_{\perp}^2(\tau_2\nabla)\,\tau_2\right\}; \tag{17b}$$

$$D_2 \equiv v_{\perp}\operatorname{div}\tau_2 - \frac{\partial}{\partial v_{\parallel}}v_{\parallel}\left\{(\tau\tau_2') + \tau(\tau_2\nabla)\,\mathrm{v}_F\right\} - \\ - \frac{1}{v_{\perp}}\frac{\partial}{\partial v_{\perp}}v_{\perp}\tau_2(v_{\parallel}\tau' + \mathrm{v}_F') + \frac{\tau_2}{v_{\perp}}\left\{v_{\parallel}\tau' + \mathrm{v}_F' + v_{\perp}^2(\tau_1\nabla)\,\tau_1\right\} \tag{17c}$$

The explicit form of the operators D_3, D_4, $A_i (i = 0, 1, 2, 3, 4)$ can be determined if all of the omitted terms are taken into account in formulas (16).

If it is assumed that $\frac{w}{wB} << 1$ and $\frac{rB}{L} << 1$ (w is the frequency of the process; r_B is the mean Larmor radius of the particles; L is the characteristic length of the inhomogeneity in the distribution of the density and fields), then the solution of equation (17) can be sought in the form of expansion in inverse powers of w_B

$$f = f_0 + \frac{1}{\omega_B} f_1 + \ldots \tag{18}$$

In the zero order approximation, we have

$$\frac{\partial f_0}{\partial \Theta} = 0, \tag{19}$$

i.e.,

$$f_0 = f_0(t, \mathrm{r}, v_{\parallel}, v_{\perp}).$$

The function f_1 must be determined from the equation

$$\frac{\partial f_1}{\partial \Theta} = \{\cos\Theta D_1 + \sin\Theta D_2 + \cos 2\Theta D_3 + \sin 2\Theta D_4\} f_0 + D_0 f_0. \tag{20}$$

In terms of the physical meaning of the problem, the quantity f_1 must be a periodic function of the angle Θ. Integrating the latter equation with respect to Θ from 0 to 2π, we obtain the necessary condition as follows:

$$D_0 f_0 = 0, \tag{21}$$

Integrating equation (20) while observing condition (21), we find /10

$$f_1 = \{\sin\Theta D_1 - \cos\Theta D_2\} f_0 + \frac{1}{2}\{\sin 2\Theta D_3 - \cos 2\Theta D_4\} f_0 + G_1 \tag{22}$$

[$G_1 = G_1(t, \mathrm{r}, v_{\parallel}, v_{\perp})$ is a function which is independent of Θ. Substituting f_1 in the kinetic equation (17), we obtain an equation which determine f_2:

$$\left\{D_0 + A_0 \frac{\partial}{\partial\Theta} + \cos\Theta\left(D_1 + A\frac{\partial}{\partial\Theta}\right) + \sin\Theta\left(D_2 + A_2\frac{\partial}{\partial\Theta}\right) + \right.$$
$$\left. + \cos 2\Theta\left(D_3 + A_3\frac{\partial}{\partial\Theta}\right) + \sin 2\Theta\left(D_4 + A_4\frac{\partial}{\partial\Theta}\right)\right\} f_1 = \frac{\partial f_2}{\partial\Theta} \tag{23}$$

(f_2 is a periodic function of Θ). Integrating equation (23) with respect to Θ from 0 to 2π and equating the result which is obtained to zero, we find the equation for determining the function G_1:

$$D_0 G_1 + \frac{1}{2\omega_B}\left\{D_2 D_1 - D_1 D_2 = \frac{1}{2}(D_4 D_3 - D_3 D_4) + A_1 D_1 + A_2 D_2 + \right.$$
$$\left. + \frac{1}{2}(A_3 D_3 + A_4 D_4)\right\} f_0 = 0. \tag{24}$$

In the general case it should be possible to show that the equation for the function G_n is obtained as the condition of periodicity of the $(n-1)^{\text{th}}$ approximation. In practice, the calculations

become very cumbersome in determining the function G_1 . Fortunately, in a certain class of problems it is possible to confine oneself to the determination of f_0 and the part of f_1 which is periodic in θ.

Let us turn to the physical meaning of the results which are obtained. Equation (21) was obtained from expression (20) as the result of integrating with respect to θ . This operation is equivalent to averaging the equations of motion of the particle (16) over the azimuthal angle θ . The averaging implies changing to the so-called drift approximation (or to the approximation of guiding centers). Using $v_{||}$ and $v_{\perp}$ to designate the values averaged over θ, we can write the kinetic equation in the drift approximation (21) in the form of the Liouville theorem in phase space t, r, $v_{||}$, $v_{\perp}$ (Ref. 5):

$$\frac{\partial f_0}{\partial t} + (\mathbf{V}_c\nabla) f_0 + \frac{dv_{\parallel}}{dt}\frac{\partial f_0}{\partial v_{\parallel}} + \frac{dv_{\perp}}{dt}\frac{\partial f_0}{\partial v_{\perp}} = 0, \quad \mathbf{V}_c = \mathbf{v}_F + v_{\parallel}\boldsymbol{\tau}, \tag{25}$$

$$\frac{dv_{\parallel}}{dt} = (\boldsymbol{\tau}, \mathbf{F} - \mathbf{v}_F') + \frac{v_{\perp}^2}{2}\operatorname{div}\boldsymbol{\tau}, \tag{25a}$$

$$\frac{dv_{\perp}}{dt} = -\frac{v_{\parallel}v_{\perp}}{2}\operatorname{div}\boldsymbol{\tau} - \frac{v_{\perp}}{2}\operatorname{div}v_F + \frac{v_{\perp}}{2}(\boldsymbol{\tau}(\boldsymbol{\tau}\nabla)\mathbf{v}_F). \tag{25b}$$

Comparing formulas (25a) and (25b) with the equations for drift motion of the particles (Ref. 5), we can see that $\mathbf{V}_c$ is the velocity of motion of the guiding center, and equations (25a, b) determine the parameters $v_{||}$ and $v_{\perp}$. In accordance with the conclusion set forth above, the terms $\sim\frac{1}{\omega_B}$ are omitted everywhere and are of a higher order of smallness. The distribution function itself, which is independent of the distribution of the particle velocity in azimuth θ, must be interpreted as a distribution function of the guiding centers. /11

In an examination of plasma dynamics, the kinetic equations must be solved together with the Maxwell equations. Difficulties arise in describing the expressions for the electrical current and the density of the charge. The crux of the matter is that *the mean macroscopic velocity* of the particles of a given type does not agree with *the mean velocity of their guiding centers*. The true current density is determined by the mean macroscopic velocities of the electrons and ions, which are expressed by the distribution functions of the particles f, and not f_0. In order to obtain the correct expressions for the mean velocities of particles of a given type, it is necessary to take into account the term of the first order in the expansion of the solution of the kinetic equation (17) in powers of $\frac{1}{\omega_B}$.

4. Let us calculate the mean macroscopic velocity of a given

type of particle. According to the definition, we have

$$n\bar{\mathbf{V}} = \int \mathbf{v}\,(f_0 + f_1)\,d\mathbf{v},\quad d\mathbf{v} = v_\perp\,dv_\perp\,dv_\parallel\,d\Theta. \tag{26}$$

Substituting the expression for f_1 from equation (22) and taking into account the fact that

$$\mathbf{v} = \mathbf{v}_F + v_\parallel \boldsymbol{\tau} + v_\perp(\boldsymbol{\tau}_1 \cos\Theta + \boldsymbol{\tau}_2 \sin\Theta),$$

we obtain

$$\begin{aligned} n\bar{\mathbf{V}} &= \int (\mathbf{v}_F + v_\parallel \boldsymbol{\tau})\left(f_0 + \frac{1}{\omega_B} G_1\right) d\mathbf{v} + \frac{1}{\omega_B}\int v_\perp\,(\boldsymbol{\tau}_2 D_1 - \boldsymbol{\tau}_1 D_2)\,f_0\,d\mathbf{v} = \\ &= n\,(\mathbf{v}_F + u\boldsymbol{\tau}) + \frac{1}{2\omega_B}\int v_\perp\,(\boldsymbol{\tau}_2 D_1 - \boldsymbol{\tau}_1 D_2)\,f_0\,d\mathbf{v} + \\ &\quad + \frac{1}{\omega_B}\int (\mathbf{v}_F + v_\parallel \boldsymbol{\tau})\,G_1\,d\mathbf{v}. \end{aligned} \tag{27}$$

It is taken into account here that $\int f_0 d\mathbf{v} = n$, $\int v_\parallel f d\mathbf{v} = nu$, where u is the mean macroscopic velocity of the particles along the field. In practice, calculation of the function G_1, and consequently the integral $\frac{1}{\omega_B}\int (\mathbf{v}_F + v_\parallel \boldsymbol{\tau}) G_1 d\mathbf{v}$, is very complex. In order to overcome this difficulty, we shall assume that the quantity $v_\parallel$ is on the order of $\frac{1}{\omega_B}$. In order to do this, it is necessary that the component of the electric field, which is parallel to B, is small ($E_\tau \ll E$). Such an assumption is usually made in the drift theory and means that the particles move primarily in a plane which is perpendicular to B. If this assumption is correct, then $\frac{1}{\omega_B}\int v_\parallel \tau G_1 d\mathbf{v}$ will be on the order of $\frac{1}{\omega_B^2}$, and it can then be disregarded. The quantity $\frac{1}{\omega_B}\int \mathbf{v}_F G_1 d\mathbf{v}$ can also be disregarded, because it is $\frac{1}{\omega_B}$ times less than $n\mathbf{v}_F$. The second integral in equation (27) determines the mean velocity acquired by the particles in a direction perpendicular to the magnetic field, due to the remaining drifts which are not included in $n\mathbf{v}_F$. The expressions for the operators D_1 and D_2 are taken from formulas (17a) and (17b). In order to eliminate the vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, it is necessary to proceed in a manner similar to that which is followed in the works (Ref. 4) and (Ref. 6). Carrying out the integration with respect to $\mathbf{v}$ and taking the fact into account that

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$$p_\perp = \frac{mn\bar{v}_\perp^2}{2} = \frac{m}{2}\int v_\perp^2 f_0\,d\mathbf{v},$$

$$p_\parallel = mn\,(\bar{v}_\parallel^2 - u^2) = m\int (v_\parallel - u)^2 f_0\,d\mathbf{v},$$

we obtain

$$\overline{\mathbf{V}} = u\boldsymbol{\tau} + \mathbf{v}_F - \frac{c}{en}\operatorname{rot}\left(\frac{p_\perp}{B}\boldsymbol{\tau}\right) + \\ + \frac{1}{\omega_B}\left[\boldsymbol{\tau},\ u\frac{d\boldsymbol{\tau}}{dt} + \frac{d\mathbf{v}_F}{dt} + \frac{p_\parallel}{nm}(\boldsymbol{\tau}\nabla)\boldsymbol{\tau} + \frac{p_\perp}{nm}\frac{\nabla B}{B}\right]. \tag{28}$$

here

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (u\boldsymbol{\tau} + \mathbf{v}_F, \nabla).$$

This expression can be readily written in another equivalent form

$$\overline{\mathbf{V}} = u\boldsymbol{\tau} + \mathbf{v}_F + \frac{1}{\omega_B}\left[\boldsymbol{\tau},\ u\frac{d\boldsymbol{\tau}}{dt} + \frac{d\mathbf{v}_F}{dt} + \frac{(p_\parallel - p_\perp)}{mn}(\boldsymbol{\tau}\nabla)\boldsymbol{\tau} + \right. \\ \left. + \frac{1}{mn}\nabla p_\perp\right]. \tag{29}$$

In order to do this, it is necessary to represent the third term of equation (28) as

$$\frac{c}{en}\operatorname{rot}\left(\frac{p_\perp\boldsymbol{\tau}}{B}\right) = \frac{p_\perp}{m\omega_B}[\boldsymbol{\tau},\ (\boldsymbol{\tau}\nabla)\boldsymbol{\tau}] + \frac{p_\perp}{m\omega_B}\boldsymbol{\tau}(\boldsymbol{\tau}\operatorname{rot}\boldsymbol{\tau}) - \\ - \frac{c}{en}\left[\boldsymbol{\tau},\ \frac{\nabla p_\perp}{B}\right].$$

The second term of the right side of the equation described determines an additional contribution, on the order of $\frac{1}{\omega_B}$, to the velocity component parallel to the magnetic field. This can be combined with the mean velocity along the field, assuming

$$u_{\text{new}} = u + \frac{p_\perp}{m\omega_B}(\boldsymbol{\tau}\operatorname{rot}\boldsymbol{\tau}),$$

as is usually done in the theory of drift motion of particles.

The meaning of separate terms in equation (28) can be clarified in the following way: $u\boldsymbol{\tau}$ is the ordered macroscopic velocity of the particles along the field; $\mathbf{v}_F$ is the velocity of electric drift, which is regarded as dominant, if the electric field is sufficiently strong. The last term in formula (28) gives the averaged drift of the particles due to the gradient of the magnetic field, of its curvature (centrifugal drift), and also due to the inertia force $u\frac{d\boldsymbol{\tau}}{dt} + \frac{d\mathbf{v}_F}{dt}$, since the examination was carried out in the moving, local system of coordinates [a comparison can be made with the corresponding expressions for particle motion in the work (Ref. 4)]. In order to clarify the role of the remaining term $\frac{c}{en}\operatorname{rot}\left(p_\perp\frac{\boldsymbol{\tau}}{B}\right)$, we can write the expression for the electric current in a plasma; /13

$$\mathbf{j} = \mathbf{j}_e + \mathbf{j}_i = en(\bar{\mathbf{V}}_i - \bar{\mathbf{V}}_e) = -c\,\mathrm{rot}\left(\frac{p_\perp}{B}\boldsymbol{\tau}\right) + \tag{30}$$
$$+ en(u_i - u_e)\boldsymbol{\tau} + \mathbf{j}_{dr},$$
$$\mathbf{j}_{dr} = \frac{c}{B}\left[\boldsymbol{\tau}, \sum_{a=e,i} m_a n\left(n_a \frac{d\boldsymbol{\tau}}{dt} + \frac{d\mathbf{v}_F}{dt}\right) + p_\parallel(\boldsymbol{\tau}\nabla)\boldsymbol{\tau} + p_\perp \frac{\nabla \mathbf{B}}{B}\right]. \tag{31}$$

In formulas (30) and (31) $p_\perp = p_{\perp i} + p_{\perp e}$, $p_\parallel = p_{\parallel e} + p_{\parallel i}$. The quantity $\mathbf{j}_{dr} + en(u_i - u_e)\tau$ is the current caused by the relative motion of the guiding centers of the electrons and ions. The Maxwell equation $\mathrm{rot}\,\mathbf{B} = \frac{4\pi}{c}\mathbf{j}$ can be written in the following form

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi e}{c}(u_i - u_e)\boldsymbol{\tau} + \frac{4\pi}{c}\mathbf{j}_{dr}, \tag{32}$$

where

$$\mathbf{H} = \mathbf{B} + 4\pi\frac{p_\perp \boldsymbol{\tau}}{B} = \mathbf{B}\left(1 + \frac{4\pi p_\perp}{B^2}\right).$$

In equation (30) $\mathrm{rot}\,\frac{cp_\perp}{B}\tau$ is the magnetization current and the quantity $\varkappa = 1 + \frac{4\pi p_\perp}{B^2}$ can be treated as the magnetic permeability of the plasma. The magnetization of a unit volume is characterized by the quantity $\bar{\mu} = \frac{p_\perp}{B^2}$. The magnetic moment of a particle moving in a Larmor circle equals $\mu = \frac{mv_\perp^2}{2B}$. Consequently $\bar{\mu}$ is the mean magnetic moment of a unit volume of plasma. The isolation of the magnetic field strength H in an examination of plasma dynamics is not customary; usually the total current j and the magnetic induction B are examined.

5. The kinetic equation (21), described for electrons and ions, together with the Maxwell equations [the expression for the current must be taken from the relationship (30)], provides the complete system of equations for describing the motion of a plasma with frequencies below the Larmor frequency of ions. Frequently, instead of kinetic equations, it is simpler to use the equations of hydrodynamics. In order to obtain the hydrodynamic equations, we followed the customary scheme, i.e., we multiplied the kinetic equation (21) by 1, /14

$v_{\|}$, $v_{\perp}^2$, $(v_{\|} - u)^2$, and integrated with respect to velocity. As a result, we obtain the equation of continuity for a given type of particles (omitting the index α)

$$\frac{\partial n}{\partial t} + \operatorname{div} n(\mathbf{v}_F + \tau u) = 0. \tag{33}$$

The equation of motion in the direction of the magnetic field is

$$\frac{du}{dt} = \left(\tau, -\frac{\nabla p_{\|}}{mn} + \mathbf{F} - \frac{d\mathbf{v}_F}{dt}\right) + (p_{\perp} - p_{\|}) \operatorname{div} \tau \tag{34}$$

and the equation of state is

$$\frac{dp_{\perp}}{dt} + 2p_{\perp} \operatorname{div} \mathbf{v}_F - p_{\perp}(\tau(\tau\nabla)\mathbf{v}_F) + \\ + p_{\perp} \operatorname{div} \tau u + p_{\perp} u \operatorname{div} \tau = -\operatorname{div} \tau q - q \operatorname{div} \tau, \tag{35}$$

$$\frac{dp_{\|}}{dt} + 3p_{\|}(\tau\nabla)u + up_{\|} \operatorname{div} \tau + p_{\|} \operatorname{div} \mathbf{v}_F + \\ + 2p_{\|}(\tau(\tau\nabla)\mathbf{v}_F) = -\operatorname{div} \tau s. \tag{36}$$

The system of equations (33) - (36) was first obtained by Chu, Gol'dberger, and Lou (Ref. 7). Equations (35) and (36) lead to the form of the equations (9) - (10), if $\mathbf{V}$ designates the quantity $u\tau + \mathbf{v}_F$. The quantities q and s represent the thermal currents (third moments of the distribution function):

$$q = \int v_{\perp}^2 (v_{\|} - u) f_0 \, dv, \quad s = \int (v_{\|} - u)^2 f_0 \, dv.$$

The Maxwell equations must be added for a complete description of the plasma.

The hydrodynamic equation (31) determines the speed of motion of particles along the magnetic field; the velocity in a perpendicular direction can be found by formula (28). The system of equations (33-36) is open. Obtaining equations for the third moments of the distribution function does not improve our position, since fourth moments enter into it, etc. The system of equations can be usually changed into a closed system by equating the quantities q and s to zero. In other words, a definite symmetry in the distribution function with respect to the longitudinal velocities is assumed. This also means that there are no currents of heat along the lines of force. This assumption is reasonable in physical terms, because the particles move much more freely along the field than they do across it, and the rotation of the particles in the Larmor circles does not lead to a transfer of energy. The magnetic field effectively "hydrodynamizes" the system, while the Larmor radius r_B replaces the free path in

the sense that, for $r_B = 0$, streams of particles and energy perpendicular to the magnetic field are absent, and, to obtain them, terms on the order of r_B/L and higher are necessary (L is the characteristic dimension of inhomogeneity given above in the distribution of the quantities being examined).

If it is assumed that $q = s = 0$, then equations (35) and (36) can be written in the form /15

$$\frac{d}{dt}\left(\frac{p_\perp}{uB}\right) = 0, \tag{37}$$

$$\frac{d}{dt}\left(\frac{p_\parallel B^2}{n^3}\right) = 0. \tag{38}$$

Thus, the equation

$$-u \operatorname{div} \tau - \operatorname{div} \mathbf{v}_F + (\tau(\tau\nabla)\mathbf{v}_F) = \frac{1}{B}\frac{dB}{dt}$$

and the equation of continuity must be used.

6. The system of equations (33) - (36) describes the dynamics of two "liquids" - electrons and ions (the index $\alpha = e, i$ should be introduced in all the quantities). The equation of motion (33) does not have a true hydrodynamic form, because it only holds for motion along the magnetic field. Let us show that the system (33) - (36) is equivalent to the system of equations of magnetic hydrodynamics with an anisotropic pressure and ideal conductivity. If the motion of electrons is not of special interest, considering the plasma to be quasi-neutral, then it is more convenient to examine the motion of the center of inertia of an elementary plasma volume, which practically coincides with the motion of ions, and the motion of electrons with respect to the ions (Ohm's law). In order to obtain the equation of motion for a quasi-neutral plasma, we should note that the electric current j can be represented as follows:

$$\mathbf{j} = \mathbf{j}_\parallel + \mathbf{j}_\perp = \tau e(u_i - u_e) + \frac{c}{B}\left[\tau, \sum_{\alpha=e,i} m_\alpha n\left(u_\alpha \frac{d\tau}{dt} + \frac{d\mathbf{v}_F}{dt}\right) + \nabla p_\perp + (p_\parallel - p_\perp)(\tau\nabla)\tau\right]. \tag{39}$$

In addition, multiplying equation $\operatorname{rot} B = \frac{4\pi}{c} j$ in a vectorial manner by B, we substitute the expression for j from equation (39). Expanding the double vector product, we obtain

$$\sum_{\alpha=e,i} m_\alpha n \frac{d}{dt}(\mathbf{v}_F + \tau u_\alpha) = -\nabla p_\perp + (p_\perp - p_\parallel)(\tau\nabla)\tau + \tau \operatorname{div} \tau (p_\perp - p_\parallel) + \frac{1}{4\pi}[\operatorname{rot} \mathbf{B}, \mathbf{B}]. \tag{40}$$

If the tensor P is introduced with the components $p_{ik} = p_{\parallel}\tau_i\tau_k + p_{\perp}(\delta_{ik} - \tau_i\tau_k)$, then - directly differentiating - it can be found that

$$\operatorname{div}\mathbf{P} = \nabla p_{\perp} + (p_{\parallel} - p_{\perp})(\boldsymbol{\tau}\nabla)\boldsymbol{\tau} + \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau}(p_{\parallel} - p_{\perp}). \tag{41}$$

Disregarding the mass of the electrons in the left side of equation (40), and introducing the term $\mathbf{V} = \mathbf{v}_F + \tau u_i$, we obtain

$$m_i n\frac{d\mathbf{V}}{dt} = -\operatorname{div}\mathbf{P} + \frac{1}{4\pi}[\operatorname{rot}\mathbf{B},\ \mathbf{B}]. \tag{42}$$

This is an equation of motion for a quasi-neutral plasma with a non-isotropic pressure tensor. /16

If the terms on the order of $\frac{1}{\omega_B}$ are disregarded, the velocity of the plasma is $\mathbf{V} = \frac{c[\mathbf{EB}]}{B^2}$. Formulating the vector products of both parts of the described equation with the vector B, and making use of the assumption that the longitudinal component of the electric field is small, we obtain

$$\mathbf{E} = \frac{1}{c}[\mathbf{VB}]. \tag{43}$$

The electric field E, consequently, is determined from Ohm's law with ideal conductivity. Substituting E from formula (43) in the Maxwell equation $\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$, we obtain

$$\frac{\partial\mathbf{B}}{\partial t} - \operatorname{rot}[\mathbf{VB}] = 0, \tag{44}$$

which designates the "freezing" of the magnetic lines of force in the case being examined. If the currents of heat are disregarded, the equations of state can be described in a form which is analogous to the equations (35), (37) or (37), (38):

$$\frac{d}{dt}\left(\frac{p_{\perp}}{nB}\right) = 0, \tag{45}$$

$$\frac{d}{dt}\left(\frac{p_{\parallel}B^2}{n^3}\right) = 0 \tag{46}$$

or,

$$\frac{dp_{\parallel}}{dt} + p_{\parallel}\operatorname{div}\mathbf{V} + 2p_{\parallel}\boldsymbol{\tau}(\boldsymbol{\tau}\nabla)\mathbf{V} = 0, \tag{47}$$

$$\frac{dp_{\perp}}{dt} + 2p_{\perp}\operatorname{div}\mathbf{V} - p_{\perp}\boldsymbol{\tau}(\boldsymbol{\tau}\nabla)\mathbf{V} = 0. \tag{48}$$

Here $p_{\perp} = p_{\perp e} + p_{\perp i}$, $p_{\parallel} = p_{\parallel e} + p_{\parallel i}$. We should note that equations

(45) and (46) can be interpreted in the following way. The quantities B and $p_\perp$ are not changed if the plasma is compressed in the direction of the magnetic field. The quantities $p_\parallel$ and n are related to the adiabatic law with $\gamma = 3$, in agreement with the fact that the energy of one longitudinal degree of freedom is increased. If the plasma is compressed in a direction perpendicular to the magnetic field, $p_\parallel$ remains constant. According to the freezing condition $B \sim n$. Consequently, equation (48) can be interpreted as an adiabatic curve with $\gamma = 2$, which points to an increase in the energy of two perpendicular degrees of freedom.

The system of equations (42-46), which is analogous to the equations of magnetic hydrodynamics, describes the behavior of a plasma in a strong magnetic field. We should point out that the absence of collisions plays a definite role in retaining the anisotropic pressure tensor throughout the entire process. The fact that this is correct /17
for each problem must be verified.

In conclusion, we should note one fact which should be kept in mind in using the system of equations (42-46) for describing equilibrium plasma configurations. Strictly speaking, the averaging method which is used in deriving drift equations for a particle [and, consequently, the kinetic equation in a drift approximation (21)], is correct if the number of transits of each particle through this system is less than L/r_B (Ref. 4). If this system exists for a sufficiently long time and this condition is disturbed, then the conclusions - obtained on the basis of the kinetic equation (21) or the hydrodynamic system (42-46) - can be considered only as approximate.

7. By way of an example of the applicability of the equations obtained, let us examine the problem of small fluctuations of a non-isothermic plasma. The solution of system (42-46) will be sought in the form $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$,

$$p_\parallel = p_{\parallel 0} + p_{\parallel 1}, \quad p_\perp = p_{\perp 0} + p_{\perp 1}, \quad \varrho = \varrho_0 + \varrho_1, \quad \mathbf{V} = \mathbf{V}_1.$$

Let us introduce the terms

$$\tau_0 = \mathbf{B}_0/B_0, \quad \mathbf{b}_1 = \mathbf{B}_1/B_0, \quad \xi = \int^t \mathbf{V}_1\, dt, \quad V_A = B_0/\sqrt{4\pi\varrho_0}.$$

If the terms of the second order of smallness with respect to the disturbances are disregarded, the system (42-46) assumes the form

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\varrho_0}\operatorname{div}\mathbf{P}_1 - V_A^2\{\nabla(\tau_0 \mathbf{b}_1) - (\tau_0\nabla)\mathbf{b}_1\}, \tag{49}$$

$$\mathbf{b}_1 = (\tau_0\nabla)\xi - \tau_0 \operatorname{div}\xi, \tag{50}$$

$$p_{\perp 1} = -2p_{\perp 0}\operatorname{div}\xi + p_{\perp 0}(\tau_0\nabla)(\tau_0\xi), \tag{51}$$

$$p_{\parallel 1} = -p_{\parallel 0}\operatorname{div}\xi - 2p_{\parallel 0}(\tau_0\nabla)(\tau_0\xi). \tag{52}$$

With the help of equations (50-52), we obtain from formula (41)

$$\operatorname{div}\mathbf{P}_1 = -2p_{\perp 0}\operatorname{div}\xi + p_{\perp 0}\nabla(\tau_0\nabla)(\tau_0\xi) + (p_{\parallel 0} - p_{\perp 0})(\tau_0\nabla)^2\xi + \\ + (p_{\perp 0} - 4p_{\parallel 0})\tau_0(\tau_0\nabla)^2(\xi\tau_0) + p_{\perp 0}\tau_0(\tau_0\nabla)\operatorname{div}\xi. \tag{53}$$

From equation (49) we find that

$$\left(\frac{d^2\xi}{dt^2}\tau_0\right) = p_{\perp 0}(\tau_0\nabla)\operatorname{div}\xi + (-p_{\perp 0} + 3p_{\parallel 0})(\tau_0\nabla)^2(\xi\tau_0). \tag{54}$$

If we take the divergence of both parts of equation (49), differentiating it twice in time, and if we express the quantity $\left(\frac{d^2\xi}{dt^2}\tau_0\right)$ with the aid of formula (54), then we obtain, noting that $\operatorname{div}\xi = -\frac{\rho_1}{\rho_0}$

$$\frac{\partial^4\rho}{\partial t^4} - \left(2\frac{p_{\perp 0}}{\rho_0} + V_A^2\right)\Delta\frac{\partial^2\rho_1}{\partial t^2} - \left(2\frac{p_{\parallel 0}}{\rho_0} - \frac{p_{\perp 0}}{\rho_0}\right)(\tau_0\nabla)^2\frac{\partial\rho_1}{\partial t^2} - \\ - \left\{\frac{p_{\perp 0}^2}{\rho_0^2} - 3\frac{p_{\parallel 0}}{\rho_0}\left(2\frac{p_{\perp 0}}{\rho_0} + V_A^2\right)\right\}\Delta(\tau_0\nabla)^2\rho_1 + \left\{3\frac{p_{0\parallel}^2}{\rho_0^2} + \right. \\ \left. + \frac{p_{\perp 0}}{\rho_0}\left(3\frac{p_{\parallel 0}}{\rho_0} - \frac{p_{\perp 0}}{\rho_0}\right)\right\}(\tau_0\nabla)^4\rho_1 = 0. \tag{55}$$

If it is assumed that the solution of the described equation is proportional to $e^{ikr+i\omega t}$, we obtain the dispersion equation (let us assume that the magnetic field is directed along the z-axis)

$$\omega^4 - A\omega^2 - B = 0; \\ A = \left(\frac{2p_{\perp 0}}{\rho_0} + V_A^2\right)k^2 + \frac{1}{\rho_0}(2p_{\parallel 0} - p_{\perp 0})k_z^2; \\ B = \left\{\frac{p_{\perp 0}^2}{\rho_0^2} - 3\frac{p_{\parallel 0}}{\rho_0}\left(2\frac{p_{\perp 0}}{\rho_0} + V_A^2\right)\right\}k^2k_z - \\ - \frac{1}{\rho_0^2}\{3p_{\parallel 0}^2 + p_{\perp 0}(3p_{\parallel 0} - p_{\perp 0})\}k_z^4. \tag{56}$$

From this equation it is possible to obtain relationships which are analogous to those which are obtained for an isotropic plasma.

Thus, for a wave which is propagated along the B_0 field we obtain

$$\omega_1^2 = 3\frac{p_{\parallel 0}}{\varrho_0}k^2, \quad \omega_2^2 = V_A^2 k^2 .$$

The first of these solutions corresponds to "one-dimensional" sound with an adiabatic curve $\gamma = 3$; the second corresponds to an Alfvén wave.

For a wave which is propagated perpendicularly to the direction of the magnetic field, we obtain

$$\omega_1^2 = 2\frac{p_{\perp 0}}{\varrho_0} + V_A^2,$$

which corresponds to an accelerated magneto-sound wave with a "two-dimensional" adiabatic exponent $\gamma = 2$. Without anisotropy of pressure, there is no slow magneto-sound wave in the direction perpendicular to the magnetic field, i.e., $\omega_2 = 0$. In the presence of pressure anisotropy, the quantity ω_2^2 can become negative, which corresponds to the appearance of instability. In the case of almost perpendicular propagation, when $\frac{k_z}{k} \ll 1$, rather large anisotropy is required for the appearance of instability /19

$$p_{\perp 0} > 3p_{\parallel 0} + \sqrt{9p_{\parallel 0}^2 + 3p_{\parallel 0}V_A^2\varrho_0}.$$

In a slow magneto-sound wave, the magnetic field is increased at the locations where the density is decreased. With sufficiently large anisotropy due to plasma diamagnetism, slow magneto-sound waves can swing spontaneously due to an increase of B at the locations where ρ decreases in a random manner. This instability, which was obtained in the work (Ref. 9), is called diamagnetic.

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COLLECTIVE PROCESSES AND SHOCK WAVES IN A RAREFIED PLASMA /20

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1. Collective Processes in a Plasma.

It is known that the relaxation processes in a rarefied plasma, in a state which is far from thermodynamic equilibrium, are as a rule accompanied by excitation of collective plasma fluctuations (for example, due to instability). The formation of such fluctuations substantially affects the transfer phenomenon in a plasma, which is most interesting from a practical point of view. The most important example is the so-called "anomalous diffusion" of a hot plasma in magnetic traps (Ref. 1). The different sides of this complex problem, which are related to the theories of stability of relatively small disturbances, have now been studied sufficiently, and the bases for a non-linear approach have been essentially provided. Another interesting example of collective processes is provided by shock waves in a plasma. In normal gas dynamics, the width of the front of the shock waves has a lower limit by a magnitude on the order of the mean free path of the molecules in a gas. Due to the "collective" properties in a plasma, the existence of specific shock waves, with a width which is significantly lower than the length of the free path, is possible. This means that even a strongly-rarefied plasma resembles, not a "Knudsen" gas, but rather a gas-dynamic medium.

The purpose of this article is to graphically present basic ideas and results of the theory of collective processes in a rarefied plasma. Primary attention is given to the qualitative aspect of this problem - to clarifying the physical meaning by an examination of different approximate models.

1. The interaction between particles in a plasma, due to the action of the electric forces at great distances, is accomplished not only by collisions, but is also the result of the influence of a so-called coordinated field. For a plasma which can be considered as an almost ideal gas, [$na^3 \gg 1$, is the condition of applicability of the "gas approximation", where n is the density of a quantity of particles; a is the Debye radius,] the kinetic theory holds, in which the /21 distribution function for the ions (electrons) $f_{i,e}(v, r, t)$ satisfies the Boltzmann-Vlasov equation

$$\frac{\partial f}{\partial t} + [H, f] = \mathrm{St}(f), \qquad (1)$$

where $[H, f]$ is the Poisson brackets; $\mathrm{St}(f)$ is the collision term.

The self-consistent field in equation (1) is calculated by including the terms containing the electric and magnetic field, which satisfy the Maxwell equations. The density of the charge and the density of the current will take the form $\rho = \sum e_k \int f_k dv$ and $j = \sum e_k \int v f dv$, where the summation is carried out for all types of charged particles present in the plasma. The collisions are taken into account in equation (1) by the collision integral $\mathrm{St}(f)$, the explicit form of which is determined by the composition of the plasma. In plasma dynamics it would seem that the "close" collisions and the self-consistent field must play roles which are essentially different. Thus, the collisions must determine the relaxation processes (establishment of the local Maxwell distribution, exchange of energy and impulse between the ions and electrons, etc.), each of which can be characterized by a certain time τ ("collision time"). The self-consistent field, on the other hand, would have to determine the "dispersion" characteristics of the plasma, the characteristics of the eigen fluctuations, and the wave properties of the plasma. For example, in the simplest case, when there is no magnetic field, the Langmuir electron frequency ω_0 $\left(\omega_0^2 = \frac{4\pi n e^2}{m}\right.$, e is the electron charge, m is the mass, n is the density$\left.\right)$ is the primary dispersion parameter. In the majority of cases in which we are interested, we shall regard the fluctuation frequency of the plasma as sufficiently large, so that $\omega\tau >> 1$, i.e.,in this plasma there are two different time scales τ and $T = \frac{2\pi}{\omega}$ (the fluctuation period, in which $T << \tau$). Therefore, in an examination of the fluctuation processes, it is possible to overlook the close collisions, assuming them to be infrequent, i.e., it is possible to exclude the collision integral from equation (1). Such an approach has received the name of "non-collision theory of a plasma", and makes it possible to simplify a study of the entire range of problems in plasma dynamics. This theory studies the processes occurring in a plasma over a period of time which is considerably less than the time of the free path τ. The original equation is the kinetic equation of Vlasov with coordinated fields without a collision integral

$$\frac{\partial f}{\partial t} + [H, f] = 0. \qquad (2)$$

As follows from the H-theorem, since entropy is retained in the absence of a collision integral, it would appear that the non-collision theory of a plasma is related to the isentropic processes, and should not describe the reverse relaxation processes of a plasma, processes for establishing thermal equilibrium ("Maxwellization"), etc. Nevertheless, it has been shown experimentally that the relaxation processes take place in a period of time which is considerably less than τ, i.e., under the conditions when the non-collision theory is used. Such anomalous dissipative properties of a rarefied plasma suggest the

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situation in the normal hydrodynamic theory of turbulence. The characteristic time of the reversible process of velocity diffusion is on the order of

$$\tau_v \sim \frac{R^2}{\nu}, \qquad (3)$$

where R is the characteristic dimension; ν is the kinematic viscosity. In actuality, the real relaxation time proves to be considerably less: the development of instability leads to turbulence - to a breaking down of the scales, and consequently to a reduction in the mixing time. Two factors play an essential role here. In the first place, the presence of a very large number of macroscopic degrees of freedom - of so-called "scales of pulsations" in the theory of turbulence - which interact with each other due to non-linear effects, by itself leads to the non-reversibility of the processes in time, with a change from the dynamic to the statistical description of the system - i.e., with a change from the Navier-Stokes equation to the equations characterizing the averaged motions[1] of the liquid.

In the second place, the role of the viscous effects increases as the energy changes to motions which are on increasingly smaller scales, due to the increase in the spatial gradients. Then the quantity R must be replaced in equation (3) by the characteristic scale[2] of pulsations l. For $l \ll R$, the diffusion velocity sharply decreases.

By determining the meaning of the analogy between the hydrodynamic turbulence and the anomalous dissipative processes in the rarefied plasma, one can discern two very important groups of similar phenomena.

1) The non-collision theory describes different plasma fluctuations and waves. Frequently, the states of the plasma prove to be unstable, so that the amplitudes of such fluctuations rapidly increase. Non-linear interactions of different types of fluctuations correspond to the interactions between scales of pulsations in hydrodynamics. A number of different fluctuations in a plasma can be regarded

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[1] For example, a system of an infinite number of coupled equations for moments of velocities.

[2] The scale of pulsation l in hydrodynamics is in every case not less than the mean free path λ.

as very large[1], and it is possible to change from a dynamic description to a statistical one. Just as in hydrodynamic turbulence, the processes become non-reversible even in a non-collision plasma. /23

2) The electric and magnetic fields of plasma fluctuations cause sharp local changes in the distribution function of the particles with respect to velocity. This is connected with the fact that every wave of the type $\exp i(\omega t - kr)$ interacts most strongly of all with so-called resonance particles - for example, with particles moving at a velocity close to the phase velocity of the wave $v \sim \frac{\omega}{k}$. As a result, large gradients are formed, but in velocity space. The collision term $\hat{D}\Delta_v f$ corresponds to the collisions between charged particles. In this term D is the "coefficient of diffusion" in velocity space which is similar to the term containing viscosity in the Navier-Stokes equation (but in velocity space). This corresponds to the case when collisions with small-angle deflections predominate - i.e., with a small change in velocity. Thus, an analogy can be made here with hydrodynamic turbulence, but a formal analogy can be made more rapidly, since viscosity in normal space (hydrodynamics) corresponds to "viscosity" in velocity space (plasma).

The theory of such anomalous phenomena in a plasma can be called the theory of collective processes, thereby emphasizing the fact that the basic role in these processes belongs to the plasma fluctuations and to the waves which represent "collective" motions of the particles in a plasma. The analogy with hydrodynamic turbulence provides an idea of the nature and scale of the difficulties encountered by the theory of collective processes in a plasma. In actuality, the situation is more complex in the theory of collective processes in a plasma, if only because the distribution function of the particles with respect to velocity depends not on four variables (r, t) - as in hydrodynamics - but on seven (r, v, t).

The basic problem in the theory of collective processes consists

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[1] Thus, the number N of degrees of freedom of the Langmuir fluctuations in a plasma can be approximated in the following way: per unit of volume the number of fluctuations

$$\frac{N}{V} \sim \int^{k_{max}} k^2 dk,$$

where k is the wave number.

As is known, for plasma fluctuations, $k_{max} \sim \frac{1}{a}$. Consequently, $N \sim \frac{v}{a^3}$, which is much greater than 1 according to the calculations.

of constructing the kinetics of unbalanced processes - processes for establishing thermal equilibrium. If the original state of the plasma is sufficiently far from an equilibrium state, then the change to the latter does not have a monotonic character, and is accompanied by an intense buildup of plasma fluctuations due to the instabilities[1]. /24

The theory of collective processes must determine the characteristic time of these transitional phenomena. The presence of sufficiently intense, unorganized fluctuations which accompany these transitional processes points to the phenomena of transfer, such as diffusion, thermoconductivity, etc. From the point-of-view of practical theory application, they are of great interest. Thus, for example, in studies on controlled thermonuclear reactions, the principle of magnetic thermal insulation of the plasma is utilized. Thus, frequently the balanced configurations of the plasma in the magnetic field prove to be unstable. The formation of instabilities can lead to great deterioration of the magnetic thermal insulation and to an increase in the streams of heat and particles at the walls as a result of the collective processes. In recent years, a great deal of experimental material has been accumulated on this subject. However, frequently the phenomena of the anomalously rapid "drift" of the plasma at the walls are not connected with specifically collective plasma processes, but with normal magnetohydrodynamic instabilities. It is difficult to draw a clear distinction between specifically plasma collective processes and turbulences of a magnetohydrodynamic nature. This is particularly true as a rarefied plasma with a very large mean free path can at times be described with very good accuracy by the equations which are similar to equations of normal magnetic hydrodynamics.

2. In many works on the dynamics of a rarefied plasma, completely different mathematical models are encountered, which can be used for describing the plasma. A very general approach to the solution of the given problem consists of using the kinetic equation with coordinated electric and magnetic fields. However, this path is rather complex. Frequently (in an examination of the problem of the theory of fluctuations and stability) "hydrodynamic" equations are used for describing the plasma (for electrons and ions individually). Strictly speaking, one cannot speak about hydrodynamics at all in the absence of collisions. Nevertheless, in many relationships this approach yields reasonable results.

For example, let us examine the question of the propagation of any wave through a plasma in the absence of a constant magnetic field.

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[1] Frequently a small deviation from the thermodynamic equilibrium is sufficient to cause the instability.

If the phase velocity satisfies $\frac{\omega}{k} >> \left(\frac{T}{M}\right)^{1/2}$, then the thermal motion of the particles is unimportant, and it can be assumed that all the ions and electrons in a given stream have the same velocities. Then it is possible to use simply the equations of motion for each type of particle. In an Euler system of coordinates, this is none other than hydrodynamics with zero temperature. If we are interested in the corrections related to small thermal scatter, then a correct result - i.e., one which coincides with the kinetic result - is obtained when we add the terms which take into account the gradients of pressure ∇p (for ions and electrons) in these "hydrodynamic" equations. We shall regard p as variable in terms of the adiabatic law, with the adiabatic exponent $\gamma = 3$. This is not surprising: in the absence of collisions, the motion with respect to each degree of freedom is independent and, as is known, for one-dimensional motion $\gamma = 3$.

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In a similar way, it is possible to justify the "hydrodynamic" simplified approach in another case. Let us assume that the phase velocity $\frac{\omega}{k}$ is considerably greater than the thermal velocity of the ions $\left(\frac{T_i}{M}\right)^{1/2}$, but considerably less than the thermal velocity of the electrons $\left(\frac{T_e}{m}\right)^{1/2}$. As was done previously, the ions can be described by equations of motion, disregarding the thermal scattering. As regards the electrons, the picture changes here. Since the electrons move much more rapidly than the waves, their electric fields - with respect to the electrons - will be quasi-static. Then, if - in the region where the electric potential φ is maximum - the electron distribution is Maxwellian in terms of velocity $f \sim \exp\left(-\frac{mv^2}{2T}\right)$, then the density of the electrons at any point will be described by the Boltsmann distribution $n = n_0 \, e^{\frac{e\varphi}{T}}$. If we now confine ourselves to the wave lengths, which considerably exceed the Debye radius a, then the electric field can be excluded from the equations, by making use of the quasi-neutrality condition: $n_i = n_e = n_0 \, e^{\frac{e\varphi}{T}}$. The term containing the electric field in the equation of motion for ions $-e\nabla\varphi$ is replaced by $-\frac{T}{n}\nabla n$. Thus, the movement of the ions conforms to the equations of hydrodynamics with $\gamma = 1$ (the isothermality is assured due to the electrons, which - moving more rapidly than the waves - succeed in equalizing the temperature). However, a hydrodynamic description does not take into account certain characteristics which are related to the presence of thermal motion. Thus, features which are caused by the presence of resonance particles - which have velocities

close to the velocity of the wave distribution - do not enter into the investigation. Such particles determine the damping of the fluctuations, which is not connected with the collisions. If $\frac{\omega}{k} >> \left(\frac{T}{m}\right)^{1/2}$ of such particles is exponentially small, the damping consequently is small [on the order of $\exp\left\{-\frac{m}{T}\left(\frac{\omega}{k}\right)^2\right\}$].

If there is a sufficiently strong magnetic field in the plasma, the situation can change. With certain limitations, the kinetic method of description, even in the absence of collisions, can lead to a hydrodynamic one. The physical reason for this consists of the fact that the magnetic field, as it were, "binds" the particles to their lines of force, and the mean macroscopic velocity of the particles will be determined "by the motion" of the lines of force themselves. Formally, such approximate equations are obtained by expanding the kinetic equation in powers of the ratio of the mean Larmor radius of the particles to the characteristic spatial scale R. The expansion with respect to $\frac{r_H}{R}$ is similar to the derivation of the normal hydrodynamic approximation of the kinetic theory, when the expansion is carried out with respect to $\frac{\lambda}{R}$ (λ is the mean free path). In fact, this expansion signifies the change to a description of the plasma as a set of quasi-particles - "Larmor circles". Two pressures are contained in the hydrodynamic equations which are thus obtained: longitudinal and transverse (with respect to the direction of the lines of force of the magnetic field). Thus, $\gamma = 2$, since the transverse motion is two-dimensional. /26

In this article, simplified hydrodynamic equations are used along with the kinetic equation in those cases when their use is valid. This considerably facilitates the examination of certain non-linear problems, if only for the reason that it makes it possible to return to the analogy with non-linear motions in normal hydrodynamics.

3. It is clear that the theory of stability plays an important role in the theory of collective processes. Ordinarily, the stability of the state of any system is studied by the method of disturbance. If the initial disturbance of a stationary state of a system increases with time, then the state is unstable with respect to the form of the disturbance. In practice, it is always a question of stability only with respect to small disturbances - to deviations from the original stationary state, when the equations describing them assume linearization, i.e., expansion with respect to the amplitude of the disturbances, and rejection of all terms of the first order in a manner similar to the theory of small fluctuations. In the theory of the stability of the state of a rarefied plasma, as a rule, many charac-

teristics of the magneto-hydrodynamic theory of stability are retained. This is related to the fact that frequently a rarefied plasma can be described very accurately - as was already noted - by magneto-hydrodynamic equations. However, unusual instabilities, which are not described within the framework of magneto-hydrodynamic equations, are inherent to a rarefied plasma. The criteria for the formation of these instabilities and the rate at which they increase can be obtained only in the kinetic theory. The collision integral is usually omitted, which is justified, if the increments for the increase in stability are considerably greater than the collision frequencies. In an examination of the instability connected with a local deviation of the plasma state from thermodynamic equilibrium, it is convenient to regard the "background" (stationary state of the plasma) as uniform and infinitely extended. A study of stability in similar cases amounts to solving the corresponding dispersion equation connecting the eigen frequency of ω with the wave vector k. Generally speaking, cumbersome calculations do not have to be carried out to obtain the criterion of instability with respect to certain simple types of disturbances; it is sufficient to confine oneself to graphic considerations (Ref. 2). The problems of the stability of "a weakly heterogeneous" plasma also yield to analysis, when it is possible to utilize the smallness of the relationship $\frac{\lambda}{R}$ (λ - is the wavelength of the disturbance; R is the characteristic dimension of "heterogeneity"). /27

Let us examine the instability with respect to wave-like distortions of the lines of force of the magnetic field. It is known that in a balanced plasma the initial distortions of the lines of force are propagated as Alfvén magneto-hydrodynamic waves, which can be graphically represented in the form of fluctuations of "elastic fibers" (the lines of force of the magnetic field). In order to clarify the instability conditions, let us examine the forces which arise with a distortion of the line of force (Figure 1). Since the particles are "bound" to the lines of force, centrifugal force arises with motion along the distorted region of the lines of force

$$F_{\text{сп}} = \int f \frac{mv_{\parallel}^2}{R}\, dv; \qquad (4)$$

this centrifugal force tends to increase the distortion.

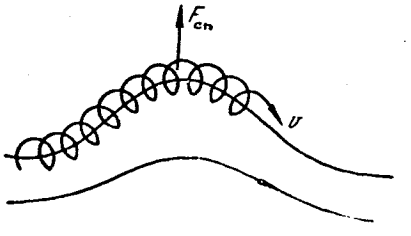

Figure 1

* Note: cn designates 'centrifugal'

In addition, since each "quasi particle" has the magnetic moment μ, which is oriented opposite the magnetic field H, a force connected to the stream of magnetization will act upon a particle in the heterogeneous magnetic field

$$j_\mu = c\nabla \times \int \mu f\, dv,$$

$$\mathbf{F}_\mu = \frac{[j_\mu \times \mathbf{H}]}{c} = [\mathrm{rot} \int \mu f\, dv \times \mathbf{H}]. \tag{5}$$

Together with the stretching force of the lines of the magnetic field, this force

$$\mathbf{F}_{st} = \frac{1}{4\pi}[\mathrm{rot}\,\mathbf{H} \times \mathbf{H}] \tag{6}$$

tends to return the line of force to an equilibrium state.

If $F_{cn} > F\mu + F_S$, the system leaves the state of equilibrium, i.e., instability arises. The following instability condition can be readily obtained from expressions (4) - (6): /28

$$p_\parallel - p_\perp > \frac{H^2}{4\pi}, \tag{7}$$

where

$$p_\parallel = \int m v_\parallel^2 f\, dv, \quad p_\perp = \int \mu H f\, dv, \quad \mu = \frac{m v_\perp^2}{2H}.$$

The velocity at which the plasma leaves the equilibrium state can be found by equating the sum of the forces $F_{cn} - F_\mu - F_S$ to the product of the mass of the plasma element by the acceleration $\frac{dv}{dt} = \frac{d}{dt}$ with $\frac{E_\sim}{H}$. It follows from the Maxwell equations that $E_\sim = H_\sim \frac{\omega}{ck}$, if the disturbance is selected in the form $\exp(i\omega t - ikx)$. Then the equation is equal to $\frac{dv}{dt} = i\omega^2 \frac{H_\sim}{kH}$. Substituting the values of the force F, we obtain

$$\omega^2 = \frac{k^2}{\varrho}\left(\frac{H^2}{4\pi} + p_\perp - p_\parallel\right). \tag{8}$$

This instability is caused by the appearance of centrifugal force arising in the motion of the particles along the distorted line of force (sometimes it is called "hose", by analogy with a rubber hose which piles up when water passes through it).

Similarly, in another limiting case ($p_\perp > p_\parallel$) we arrive at the criterion of instability which has the form

$$p_\perp \gg p_\parallel\left(1 + \frac{H^2}{8\pi p_\perp}\right). \tag{9}$$

Conditions (7) and (9) show that the plasma is unstable if the

*Note: S designates 'stretching'

distribution of the particles in terms of velocity differs, to a sufficiently great extent, from an isotropic one. With a decrease in H, instability can occur for smaller anisotropies. However, for very small H the Larmor radius of the particles is large, and the concept of quasi particles - Larmor circles - does not hold. Nevertheless, in the range $H \to 0$, anisotropic instability also exists. The instabilities being examined are aperiodic, i.e., their dependence on time has the form $\exp \gamma t$. The deviation of the plasma state from a thermodynamically-balanced state can lead to a buildup of the waves, i.e., to the appearance of fluctuating instability. The criteria for the formation of such instability - i.e., the conditions for a change in the sign of the imaginary part of ω_i of the frequency $\omega = \omega_r + i\omega_i$ - can be readily obtained by examining the balance of the energy exchange between any plasma waves, arising as a result of fluctuations, and particles of the plasma. For very small $\omega_i (\omega_i \ll \omega_r)$, a wave with a given ω and the corresponding wave vector is almost periodic. Ions (electrons) of the plasma, oscillating in a periodic field of the waves, do not change their energy on the average. Those particles having a velocity distribution, for which the condition of resonance with a wave is fulfilled, comprise an exception to this. In the absence of a magnetic field, in an undisturbed plasma only those particles are found in resonance whose velocity is close to the phase velocity of the wave $\frac{\omega}{k}$ (the resonance condition: $\omega - kv = 0$). In the presence of a constant external magnetic field, those particles will also effectively interact with a wave for which - in their own system of coordinates - the frequency of the wave $\omega' = \omega - kv$ due to the Doppler effect will be close to the cyclotron frequency $\omega_s = \frac{eH}{mc}$ (or to one of its harmonics $n\omega_s$ for $n = \pm 1, \pm 2, \ldots$) . The particles, in which the velocity component along the magnetic field satisfies this condition, will be continuously accelerated (or slowed down) by the field of the wave, in a manner similar to that in which the ions in a cyclotron are accelerated. In the simplest case, when there is no constant magnetic field, either purely transverse, or purely longitudinal, waves can be propagated in the homogeneous plasma. The transverse waves will not be examined, since their phase velocity exceeds the speed of light $\left(\varepsilon = 1 - \frac{\omega_0^2}{\omega^2}\right)$. With respect to the longitudinal Langmuir electron fluctuations, their phase velocity has a lower limit on the order of the thermal velocity of the electrons (the corresponding minimum length of the wave is on the order of a Debye radius), and it increases with an increase in the wavelength. Let us regard a Langmuir wave, with the frequency ω (and a phase velocity $\frac{\omega}{k}$) in a system of coordinates which move - with respect to the laboratory system - with the velocity $\frac{\omega}{k}$, as

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an electrostatic potential representing the fixed sinusoid of the amplitude φ_0 : an alternation of the potential "holes" and "humps", for the electrons. Those electrons whose velocity differs noticeably from $\frac{\omega}{k}$ will move freely in this periodic field, retaining their energy, on the average. The electrons, whose velocity v differs from $\frac{\omega}{k}$ by a quantity which is less than $\sqrt{\frac{2e\varphi_0}{m}}$, will be reflected from the potential "humps". These electrons can be divided into two groups: the velocity of one group exceeds $\frac{\omega}{k}$ and the others have a velocity which is less than $\frac{\omega}{k}$. The electrons of the first group, overtaking the "humps" of the potential, are reflected, passing the energy to the wave; the electrons of the second group "urge on" the wave, obtaining energy from it. A simple examination of the balance of energy in the reflection of electrons from the potential "humps" leads to the instability condition which is called inverse Landau damping. The amplitude of the wave will increase, if the energy changes entirely from the electrons to the wave. This occurs in the case /30 when the number of electrons of the first group is greater than those of the second group, i.e., if

$$\frac{df}{dv}\left(v = \frac{\omega}{k}\right) > 0. \qquad (10)$$

In order that this condition be fulfilled, it is necessary that the distribution function of the electrons with respect to velocity have at least a single secondary maximum in the region of velocities which exceed the thermal velocity. If everywhere $\frac{df}{dv} < 0$, then $\omega_i < 0$, i.e., the wave damps out (this is the Landau damping) (Ref. 3).

By approximating the work accomplished by the electric field of the wave, we can derive the instability criteria in a similar way for the cyclotron resonance $\omega = \omega_s - kv$, which is important for transversely-polarized waves which are propagated along a constant magnetic field.

4. A very difficult problem encountered in the theory of collective processes in a plasma is that dealing with what state the plasma changes into as a result of the influence of instability. The disturbances which increase exponentially sooner or later become so great that the conditions for the applicability of a linear examination are destroyed. It would appear that the principal answer to this problem can be given for fluctuating instability. In this case, the state into which the plasma changes due to the instability must apparently be an unusual

mixture of two groups: particles and waves. The interaction of the particles with the waves, in particular, leads to instability. The interaction between the waves leads to a completely non-linear effect. If any wave could "exist" for a duration of time which is considerably greater than its period $\left(t >> \frac{2\pi}{\omega}\right)$, the designation "quasi particle" can be applied to it. A group of such wave-quasi particles can be described by the corresponding distribution function of the waves with respect to the quasi impulses (wave vectors k) which satisfy the corresponding kinetic equation. In formal terms, the situation becomes very similar to that which occurs in the quantum theory of a solid body where a mixture of two gases is also examined: electronic and phononic. But the fact is that in the theory of plasma instability, everything is much more complex. Since the equations are principally non-linear, it makes sense to examine only states which are far from thermodynamic equilibrium. In order to describe the corresponding kinetic equations of a turbulent plasma theory, it is necessary to know the form of the corresponding collision terms: the wave-particle and the wave-wave. The first of these is given in the so-called "quasi linear" theory which considers the role of small non-linear effects only once - it takes into account the distortion of the particle distribution functions due to the reverse action of the waves (Ref. 4 - Ref. 6).

In a quasi-linear approximation, the particle distribution functions with respect to velocity are represented in the form of the sum of two parts: a slowly changing one $f_0(v, t)$ (we shall call it the "background") /31 and a rapidly oscillating one $f_\sim(v, t)$. The slow change in the "background" is due to the inverse influence of the fluctuations on the particles, caused by the average quadratic effects of small, rapid oscillations, similarly to the well-known method of Van der Pol in non-linear mechanics. On the other hand, the term "quasi linearity" means that the direct interaction betwwen different harmonics, which is caused by the non-linearity, is not taken into account. Therefore, the balance of energy in the k^{th} harmonics of fluctuations is determined, as in the linear theory of stability, by the equation

$$\frac{d}{dt}E_k^2 = 2\nu E_k^2, \tag{11}$$

where ν is the imaginary part of the frequency.

Let us examine the manner in which the equations of quasi linear approximation are obtained for longitudinal electron Langmuir fluctuations in the homogeneous case

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{eE}{m}\cdot\frac{\partial f}{\partial v} = 0, \quad \frac{\partial E}{\partial x} = 4\pi n_\sim e, \quad n_\sim = \int f_\sim dv. \tag{12}$$

Let us separate the distribution function into slowly - and rapidly -

changing parts

$$f_{\sim} = \sum \left(f_k e^{i(kx-\omega_k t)} + \text{complex conjugate}\right),$$
$$E = \sum \left(E_k e^{i(kx-\omega t)} + \text{complex conjugate}\right). \tag{13}$$

The quantities f_k and E_k are related to each other by the usual relationship, just as in the linear theory

$$f_k = -i \frac{e}{m} \cdot \frac{1}{\omega_k - kv} \cdot \frac{\partial f_0}{\partial v} \cdot E_k. \tag{14}$$

We obtain the equation for the slowly-changing part of the distribution function f_0 by averaging with respect to the rapid oscillations

$$\langle f \rangle = f_0. \tag{15}$$

For such averaging, it is necessary that there be many waves with different wave vectors and randomly-distributed phases simultaneously in the plasma. The wave "packets", made up of such waves, must be wide enough to make it possible to disregard the capture of particles in the "potential" holes of individual harmonics of the packet. For example, in the case being examined of longitudinal Langmuir fluctuations, it is necessary that the scattering of the phase velocities of the waves in the packet considerably exceeds the velocity at which the captured wave particle would move in the potential hole $e\varphi_0 : \Delta \frac{\omega}{k} >> \left(\frac{e\varphi_0}{m}\right)^{1/2}$. Taking the fact into account that $\langle Ef_0 \rangle = \langle E \rangle f_0$, we obtain from (12) - (14) the following equation for f_0:

$$\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} = 0, \tag{16}$$

where the diffusion coefficient in velocity space D is proportional to the square of the electric field of the waves /32

$$-\frac{e^2}{m^2} \sum_{kk'} \Big\langle \left(E_{k'} \cdot e^{i(\kappa' x - \omega_{k'} t)} + \text{complex conjugate}\right) \left(\frac{E_k}{i(\omega_k - kv)} e^{i(kx-\omega_k t)} + \right.$$
$$\left. + \text{complex conjugate}\right) \Big\rangle = -\frac{e^2}{m^2} 2\pi \sum_k |E_k|^2 \operatorname{Im} (\omega_k - kv)^{-1}.$$

Equation (16) describes the inverse effect of Langmuir fluctuations on the distribution function of the particles. The region of applicability for the equations of the quasi-linear theory is limited to those cases in which the increment (or decrement) of the fluctuations is considerably less than their frequency. If this condition is not fulfilled, the division of the distribution function into rapidly-oscillating and slowly-changing parts is impossible.

It is clear from equation (16) that for an average distribution function of the particles f_0, with excitation of the collective degrees of freedom - waves -, there is additional diffusion in the plasma in velocity space, in addition to the customary "collision" diffusion. In contrast to the original equation (12), the given equation does not retain entropy. There is nothing surprising in this, since the averaging procedure, used in changing from expression (12) to equation (16), corresponds to the change from a dynamic description to a static description of the plasma. The use of a similar approach to the description of waves leads to the kinetic equation of waves - quasi particles (Ref. 7 - Ref. 10). Such a statistical approach is essentially equivalent to the correlation method used in the theory of hydrodynamic turbulence. For wave instabilities with $\frac{\nu}{\omega} \ll 1$ (the increment is much less than the frequency), the coupled chain of equations for the correlation functions can be uncoupled with respect to the small papameter $\frac{\nu}{\omega}$ (Ref. 11). Instabilities of a non-wave character with $\nu \gtrsim \omega$ cannot be examined by this method, since there is no small parameter in the problem. In similar cases in the theory of collective processes, semi-quantitative estimates are used for obtaining the results.

The present report is devoted to the explicit use of the theory of collective processes, in order to construct a theory for the thickness of shock waves in a rarefied plasma.

Due to the "collective" properties, in a rarefied plasma the existence of specific shock waves having a thickness which is considerably less than the mean free path is possible. At first glance, such a conclusion would seem to be paradoxical. Let us represent (Figure 2) a shock wave with the thickness Δ which is considerably less than the mean free path l. It would appear that the more rapid particles ($v > u$) from the region on the left (the plasma heated by the shock wave) - freely moving to the side of the non-disturbed plasma - can spread the transitional region up to the thickness l - the mean free path. What can avert such an erosion of the transitional region? /33

1) A very simple case holds in the presence of a magnetic field which is parallel to the plane of the front. Such a magnetic field turns the ions and electrons of the plasma at distances on the order of their Larmor radius r_S. Consequently, it can be expected that $\Delta \sim r_H$. A sufficiently strong magnetic field $\left(\frac{H^2}{8\pi} \gg nT\right)$ impedes the spreading action, even if it does not necessarily lie in the plane of the front. This is related to the fact that the velocities of the shock wave for $\frac{H^2}{8\pi} \gg nT$ are considerably greater than the thermal

velocity of the particles, and consequently the portion of ions (electrons) overtaking the wave is exponentially small. The following apparent paradox arises in problems of this type. The conditions at both sides of the front of the shock wave are related to the corresponding conservation laws (Hugoniot adiabatic curve), according to which the energy of the translational motion of the non-disturbed plasma is transformed into internal energy of the plasma after the passage of the shock waves. What leads to dissipation, if $\Delta \ll l$? The answer to this question lies in the fact that, in the disturbed state of the plasma, behind the front of the wave, the basic portion of the internal energy undergoes intensive plasma fluctuations. The origin of such non-linear fluctuations is not necessarily connected with the instability of the plasma. It is closely connected with the specific dispersion properties of the plasma. The second section is devoted primarily to a discussion of the theory of non-linear, ordered fluctuations in a plasma (its results are of interest, irrespective of the connection with shock waves). The non-stationary, non-linear motions of a plasma represent a problem which is too complex (it is possible to examine only certain special cases with different simplifying assumptions). In return, the steady, non-linear, non-damped fluctuations are examined relatively completely. We should note here the useful analogy between the non-linear fluctuations of a plasma and the waves of finite amplitude on the surface of a heavy liquid in a canal of finite depth. In the theory of non-linear fluctuations of a plasma, "isolated" waves also appear, the velocity of which depends on the amplitude. The non-linear waves in a plasma can disintegrate due to different plasma instabilities. Certain types of instabilities of non-linear waves are examined in the conclusion of section 2. One of /34
possible types of instabilities - the so-called "bunched" instability - appears in non-linear waves in a magnetic field, and is connected with the electric current of the wave. If the ordered velocity of electrons, with respect to the ions, exceeds their mean thermal velocity, the energy of the waves due to the bunched instability will be transferred to the energy of the electrostatic longitudinal fluctuations of the plasma. Another type of instability exists which is inherent to non-linear periodic waves - the so-called "disintegration" instabilities, when the regular fluctuations disintegrate, producing an entire spectrum of unordered waves. This instability, in the well-known sense, is similar to the disintegration of collective disturbances in the quantum theory of systems of many particles. The combination of all these factors leads to the formation of the structure of the shock wave (Section 3).

2) When the magnetic field is small, or is not present in general, the mechanism which decreases the spreading of shock wave front is different. Let us assume that, due to the spreading, a certain portion of the rapid particles penetrates the non-disturbed plasma before the

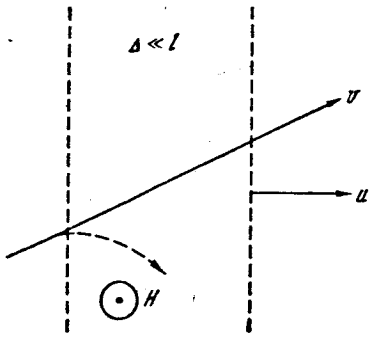


Figure 2.

front of the shock wave (see Figure 2). Then the state of the plasma in this region represents a mixture of balanced, non-disturbed distribution of the particles and a certain group of rapid particles, i.e.,it becomes unstable (ultimately, the distribution of the particles with respect to the velocities will differ from the Maxwellian). The unbalanced plasma is unstable with respect to the buildup of a different type of fluctuation. The fluctuating electric and magnetic fields of the fluctuations, formed due to the instability, lead to a scattering of the ions and electrons. In other words, in the presence of such a type of fluctuating fields, it is necessary to redefine the concept of the mean free path. In a sufficiently rarefied plasma, the scatter by the unbalanced noise is considerably more significant than the normal "pair" Coulomb particle collisions.

2. NON-LINEAR FLUCTUATIONS OF THE PLASMA

1. The most important role of the non-linear effects, as is known, consists of increasing the slope of the leading edge. However, in plasma dynamics the dispersion effects frequently begin to play an important role as the slope of the front increases. This leads to interesting characteristics in the asymptotic motions which arise - to the spontaneous formation of intense fluctuations due to the competition between the non-linearity and the dispersions. This section is devoted to the systematic presentation of the theory of non-linear, non-damped fluctuations. Let us begin the discussion of this problem with a qualitative examination of non-linear deformation of the form of any initial disturbance. In the linear theory, the fluctuating motion of /35
the plasma represents the superposition of individual, non-interacting waves - harmonics [$\exp i(\omega t - kr)$, where ω is the frequency; k is the wave vector]. Generally speaking, there is a definite connection between ω and k - the law of dispersion $\omega = \omega(k)$. It is clear that if the non-linearity of the fluctuations is taken into account, it can change the picture of the motion, which holds for the linear theory. Let us turn to a graphic analogy with sound fluctuations in normal gas dynamics. It is well-known that sound waves, which are harmonic in a linear approximation, are distorted due to the finiteness of the amplitude with the passage of time. This non-linear deformation consists of the fact that the profile sections, to which a large velocity corresponds, tend to determine the sections with a smaller velocity, and discontinuities are ultimately formed (if it is not possible to damp the sound).

Now let us trace the possible non-linear distortion of the profile of any harmonic wave in a rarefied plasma. The tendency toward increasing the slope of the leading edge due to the non-linearity is retained in waves in a rarefied plasma. (An exception to this is provided by waves with transverse polarization - such as, for example, Alfvén magnetohydrodynamic waves. Non-linear terms of the type $(\mathbf{v}\nabla)\mathbf{v}$ do not enter into the equations describing such waves). If the increase in the slope of the leading edge is limited by the dissipative effects in gas dynamics, the effects of dispersion can then play a basic role in a rarefied plasma. Using the example of an harmonic wave, we can illustrate the "competition" between non-linearity, which tends to reverse the waves, and dispersion in the following way. An increase in the slope of the leading edge means the formation of higher harmonics in the wave under the influence of non-linearity. In the first (linear) approximation, each wave is harmonic [$\exp i(\omega t - \mathrm{kr})$]; in the second approximation, the second harmonics must appear (as takes place in a sound wave). In expansion with respect to wave amplitude, the equation for correcting the second approximation takes the form

$$\hat{L}_0 f_2 = \hat{l} f_1^2 \exp i(2\omega t - 2\mathrm{kr}). \qquad (17)$$

Here f is the deviation of any quantity, characterizing the plasma or field, from the equilibrium value (the indexes 1, 2 designate, respectively, the first and second approximation); L_0 is the linear operator corresponding to the linear fluctuations of the plasma with some kind of a dispersion law $\omega = \omega(\mathrm{k})$. In the well-known sense, equation (17) has the form of an equation of motion of an "oscillator" under the influence of the driving force $\sim f_1^2$. It is clear that the second harmonics will be excited, if this force is in resonance with the eigen frequency of the oscillator, i.e., if the wave vector, which is equal to $2k$, corresponds to twice the original frequency (in the law of dispersion). This "resonance" will be accomplished only for the linear law of dispersion $\omega = ck$, as takes place in normal sound. Generally speaking, for an arbitrary law of dispersion, there will not be an energy transfer from the main harmonics to the second harmonics, etc., if "the inducing" force is far from resonance. This qualitative illustration shows that the periodic waves in a plasma, in the frequency region where the deviation from the linear law of dispersion is significant, can be propagated without distortion of their form due to non-linearity. Knowing the course of the dispersion curves $\omega(k)$ from the theory of linear fluctuations of a plasma, we can thus foresee certain general properties of non-linear movement. For example, let us deal with magnetic-sound fluctuations which are propagated across the magnetic field in a plasma. For frequencies less than ω_{H_i} - the Larmor frequency of the ions -, the phase velocity of such fluctuations equals

/36

$$\frac{\omega}{k}=\left(\frac{\partial p}{\partial \rho}\right)^{1/2}=\left(\frac{H_0^2}{4\pi\rho_0}+2\frac{p_0}{\rho_0}\right)^{1/2}, \qquad (18)$$

where H_0 is the nondisturbed magnetic field; ρ_0 is the density; p_0 is the pressure. With an increase in the frequency, the phase velocity decreases due to dispersion. In the general case, the dispersion law is very complex even for such waves. Let us examine two opposite limiting cases.

Plasma of a Low Pressure $\left(p_0 \ll \frac{H_0^2}{8\pi}\right)$.

The phase velocity of the value $\frac{H_0}{\sqrt{4\pi\rho_0}}$ for small frequencies with an increase in ω decreases up to zero for a frequency of $(\omega_{Hi}\omega_{He})^{1/2}$ (the so-called "hybrid" frequency $\left(\frac{e^2H^2}{mMc^2}\right)^{1/2}$; mM represents the electron and ion mass). The dispersion curve for this type of fluctuations is shown in Figure 3 for purposes of visualization. The corresponding dispersion law has the form

$$\frac{\omega^2}{k^2}=\frac{H_0^2}{4\pi\rho_0}\cdot\frac{\omega_0^2/c^2}{k^2+\omega_0^2c^2};\quad \omega_0^2=\frac{4\pi nl^2}{m}.$$

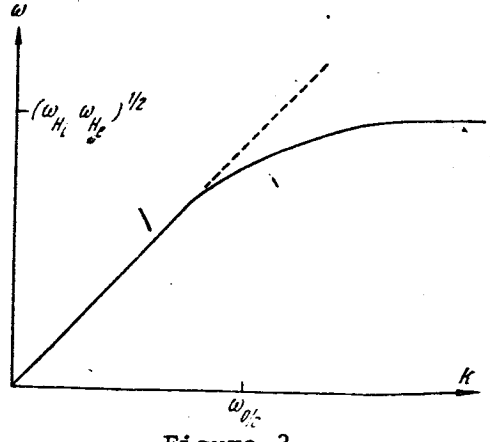


Figure 3.

Deviations from the linear dispersion law begin to appear eventually for $k \to \frac{\omega_0}{c}$. The quantity $\frac{c}{\omega_0}$ determines the characteristic spatial scale for the non-linear magneto-sound waves which are established. All of this holds for a plasma in a magnetic field which is not too strong $\frac{H^2}{8\pi} \ll nmc^2$. In such a plasma, the condition in the magneto-sound wave is quasi-neutral. /37

For example, if very strong magnetic fields are examined

$$\frac{H_0^2}{8\pi} \gg n_0 mc^2,$$

then, for frequencies greater than ω_{Hi}, the deviation from quasi-neutrality becomes significant. The law of dispersion for such fluctuations has the form (once again disregarding the thermal motion)

$$\frac{\omega^2}{k^2} = \frac{H_0^2}{4\pi\rho_0} \frac{\frac{4\pi\rho_0}{H_0^2}\Omega_0^2}{k^2 + \frac{4\pi\rho_0^2}{H_0^2}\Omega_0^2} \qquad \left(\Omega_0^2 = \frac{4\pi n l^2}{M}\right). \tag{19}$$

The phase velocity strives to zero for an ionic Langmuir frequency of $\omega \to \Omega_0$. The characteristic length, at which the deviations from the linear law of dispersion are significant, is $\frac{H_0 M}{4\pi\rho_0 e}$.

Plasma of "Large Pressure" $\left(p_0 \gtrsim \frac{H^2}{8\pi}\right)$.

In this case, dispersion begins to appear for $\omega \to \omega_{H_i}$. In fact, when the frequency ω becomes greater than ω_{H_i}, the trajectory of an ion will be slightly bent by the magnetic field during the time of one oscillation. In other words, its movement in the wave will not be two-dimensional, but one-dimensional. Therefore, the effective index of the adiabatic curve γ of the ions, which characterizes the velocity of the magneto-sound wave, for $\omega > \omega_{H_i}$ must be assumed to equal 3 (and not 2, as is the case for $\omega < \omega_{H_i}$). Consequently, the phase velocity of the wave for $\omega > \omega_{H_i}$ is increased. Thus, in "large pressure" plasma the phase velocity of the waves must not decrease, but rather increase with frequency, in the frequency interval being considered. The characteristic dimension of the non-linear waves in this case is also different; it is on the order of the Larmor radius of ions.

Up to the present we have considered waves which are propagated in a direction which is exactly perpendicular to the magnetic field. As the linear theory of small fluctuations shows, the laws of dispersion vary greatly, even when the direction in which the waves are propagated deviates little from the perpendicular direction. This is related to the fact that in such waves the component of the electric field appears along H_0. Under the influence of this electric field, the electrons along H_0 move much more rapidly than across H_0, but strongly distort the distribution of the currents and the charges in the wave. Let us again examine a "cold" plasma. For angles satisfying the condition /38 $\left(\frac{m}{M}\right)^{\frac{1}{2}} \ll \theta \ll 1$, the law of dispersion which connects ω with k acquires a particularly simple asymptotic form (if we do not limit ourselves to $\lambda \gg \frac{c}{\omega_0}\ 1/\theta$)

$$\frac{\omega^2}{k^2} \approx \frac{H_0^2}{4\pi\rho_0}\left(1 + \frac{k^2\theta^2 c^2}{\Omega_0^2}\right). \tag{20}$$

The deviation from the linear course $\omega = \omega(k)$ becomes significant

for wavelengths on the order of $\frac{c}{\Omega}\theta$. The phase velocity increases with an increase in the frequency in such waves. It can be expected that this would lead to a change in the nature of the non-linear movement.

Let us now turn to the case when the magnetic field is generally absent. Thus, as is known from the linear theory, ion sound fluctuations in a rarefied plasma display the relationship $p_e >> p_i$, if the pressure of the electrons considerably exceeds the pressure of the ions. Such conditions can be realized, for example in a nonisothermic plasma, when the temperature of the electrons considerably exceeds the ion temperature. For purposes of simplicity, if the ions are assumed to be "cold" ($T_i = 0$), then the law of dispersion will conform to the law (19):

$$\left.\begin{aligned} \frac{\omega^2}{k^2} &= \frac{T_e}{M}\cdot\frac{\varkappa^2}{\varkappa^2+k^2}, \\ \varkappa^2 &= \frac{\Omega_0^2 M}{T_e}. \end{aligned}\right\} \qquad (21)$$

The Debye radius $1/\varkappa$[1] plays a role on the characteristic scale here.

All the cases being considered show that, at small distances, the effects of dispersion become significant. In customary hydrodynamics, at small distances the dissipative effects, which limit the growth of the slope of the leading edge, become significant. In contrast to usual gas dynamics, in a rarefied plasma the dispersion effects are the cause of this. The difference between these two groups of effects finds expression in the mathematical structure of the original equations. The dissipative effects (viscosity, thermal conductivity, etc.) disturb their reversibility - they increase the order of magnitude of the derivatives by an odd number (thus, for example, in gas dynamics the viscosity adds terms with second derivatives to the Euler equation). The dispersion effects do not disturb the reversibility, and increase the order of magnitude of the differentiation in the equations by an even number.

By way of an example, let us examine the system of equations /39
which describe the extension of ionic-sound fluctuations for the condition $T_e >> T_i$. Under the given assumptions, a one-dimensional, planar motion will be described by the system of equations:

[1] Finally, the fluctuations of electrons in a cold plasma without a magnetic field have simply $\omega^2 = \omega_0^2$ as the dispersion relationship. There is no characteristic scale here, and it is possible to construct the steady non-linear waves of any spatial period.

$$\left.\begin{aligned} M\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)&=-e\frac{\partial \varphi}{\partial x};\\ \frac{\partial n_i}{\partial t}+\frac{\partial (n_i v)}{\partial x}&=0;\\ -\frac{\partial^2 \varphi}{\partial x^2}&=4\pi e\left(n_i-n_0 e^{\frac{e\varphi}{T}}\right). \end{aligned}\right\} \qquad (22)$$

Here M, v, n_i are, respectively, the mass, velocity, and density of the ions. The last equation of system (22) contains the chief (second) derivative. For motion with a characteristic scale, which considerably exceeds the Debye radius $(T/4\pi ne^2)^{\frac{1}{2}}$, it can be assumed that the plasma is quasi-neutral: $n_i = n_0 \exp(e\varphi/T)$, i.e., in the last equation of the system (22) term $\partial^2\varphi/\partial x^2$ must be discarded. Then, excluding the electric field from the equations, we obtain

$$\left.\begin{aligned} M\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)&=-\frac{T}{n}\cdot\frac{\partial n}{\partial x},\\ \frac{\partial n}{\partial t}+\frac{\partial}{\partial x}nv&=0. \end{aligned}\right\} \qquad (23)$$

With respect to its form, this system agrees with the equations of isothermic motion ($\gamma = 1$) of normal gas dynamics. Generally speaking, the front of any initial disturbance will become steeper with the passage of time. Such a non-linear distortion of the disturbance profile can be most graphically illustrated for the special case when the initial distribution of velocity and density is such that one of these quantities can be represented as the function of the other. In normal gas dynamics, for a similar case, the Rieman solution exists, which describes the so-called simple wave of arbitrary amplitude. The dependence of velocity on time and coordinates is determined in this solution by the implicit function

$$x = t[v \pm c] + \chi(v), \qquad (24)$$

where c is the speed of sound, and $\chi(v)$ is the function which is dependent on the initial conditions. According to expression (24), the flow profile will change, so that, beginning at a certain moment, the solution becomes three-valued. In normal gas dynamics (with a small mean free path), a steady flow arises with a discontinuity (shock wave) under similar conditions. On the basis of the described mathematical analogy, a hypothesis was formed regarding the possibility of a non-collision shock wave in a rarefied plasma. However, in the case /40
under consideration, due to the fact that only the leading edge of the disturbance is sufficiently steep, the influence of the dispersion effects is included [for problem (23), for example, they are related to the deviation from quasi-neutrality]. It is interesting to note that in ordinary hydrodynamics of an incompressible liquid, the analogy

between such non-linear motions - non-linear waves on the surface of a heavy liquid - has been studied in detail. If the depth of the canal is small, the equations of two-dimensional motion are reduced, as is known, to the equations of one-dimensional motion. (For a mean velocity of the liquid v in a given cross-section, and depth of the liquid h):

$$\left.\begin{aligned}\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)&=-g\frac{\partial h}{\partial x},\\ \frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hv)&=0,\end{aligned}\right\} \tag{25}$$

where g is the acceleration of gravity. These so-called "equations of shallow water" represent the zero approximation in the asymptotic expansion of exact equations of hydrodynamics, for an incompressible liquid in a canal of a finite depth with respect to the small parameter h_0/L, where h_0 is the depth of the canal, and L is the characteristic dimension (for example, of the wave length). In terms of their form, the equations for shallow water coincide with the equations describing planar isentropic flows of a compressible gas (with the adiabatic exponent $\gamma = 2$). Therefore, the Rieman simple-wave type of solutions make sense in the theory of shallow water. It follows from the solutions that the arbitrary initial form of the liquid surface tends toward the formation of ridges with the passage of time. In addition, the equations for shallow water accurately coincide with the equations of motion of a rarefied plasma (under the condition $nT \ll \frac{H^2}{8\pi}$) across the magnetic field, for wave lengths which considerably exceed c/ω_0 (if, for example, $H^2 \ll nmc^2$). Thus, the magnetic field H plays the role of the canal depth h in the plasma equations. In gas dynamics, the dissipative effects begin to play a role, as the slope of the leading edge increases. On the other hand, dispersion effects appear in the dynamics of a rarefied plasma, when the characteristic dimension approaches c/ω_0. In the theory for shallow water, when L approaches h_0, the dispersion effects also begin to play a role. Equations (25) no longer hold, but when h_0/L are still small it is possible to "improve" them by adding the additional terms of expansion with respect to h_0/L, of the exact equations. These terms have the form of the chief derivatives which are responsible for the dispersion effects. The dispersion law for waves having a small amplitude in such a theory takes the form

$$\frac{\omega^2}{k^2}=\frac{g}{k}\operatorname{th}(kh_0).$$

For small kh_0, it is possible to arrive at the form /41

$$\left(\frac{\omega}{k}\right)^2\approx gh_0\left[1-\frac{1}{3}(kh_0)^2+\cdots\right. \tag{26}$$

At the present time, non-linear fluctuations on the surface of a

heavy liquid have been studied in detail for the case of so-called steady waves, i.e., waves which do not change their form with the passage of time. In addition to periodic waves with wavelengths on the order of h_0, so-called "isolated" waves exist which represent individual, extended rises in the level of the liquid in the canal.

On the basis of the given analogy with waves in a rarefied plasma, one can expect the existence of similar periodic and isolated waves in the latter. However, due to the diversity of the dispersion laws in a plasma, we can see the most diverse types of steady waves in different cases. Thus, under definite conditions, isolated waves of "rarefaction" can be propagated throughout a plasma (to which isolated waves of the trough-type would correspond in the theory of surface waves).

In Section 3, it will be shown that in a rarefied plasma non-linear waves of such a type are directly connected with shock waves. Up until the present, the discussion has been devoted to the finite, but small, amplitude of the waves. For waves with a rather large amplitude, the situation can be completely different. The dispersion effects can be insufficient to limit the increase of the slope, and the front can be "reversed" at amplitudes exceeding a certain critical value. Regions of multiple-velocity flow are formed (it is obvious that this holds for an initial plasma).

2. In gas dynamics, a shock wave is, generally speaking, the asymptotic form (for $t \to \infty$) of any initial motion. What must be the nature of asymptotic motions in a rarefied plasma? It can be expected that for $t \to \infty$ ultimately some kind of stationary wave motion is established. Assuming that such steady motions exist, it is not difficult to study them in a one-dimensional case, by directly solving the equations of plasma dynamics. A standard procedure for obtaining the solutions consists of selecting a system of coordinates, which moves with the wave, in the original equations. In such a frame of reference, the dependence on time is missing, and the problem is reduced to determining the stationary flow. The velocity of the wave u first enters the problem as a free parameter. Then the boundaries, within which u can change - and also the connection between u and the amplitude of the wave - are determined from the solvability condition. A study of the steady, non-linear waves - in those cases when the dispersion law of corresponding small-amplitude fluctuations tends toward the linear one at large wavelengths, and the dispersion effects appear for short waves - is of interest for the theory of the shock waves thickness.

Let us begin the investigation with waves propagated across the magnetic field. In an approximation of a small Larmor radius ("drift" approximation), the hydrodynamic equations hold here. A trivial plane-parallel stream is the only type of steady motion for these equations. In order to determine the non-trivial steady motions, it is necessary to /42

take into account the dispersion effects which appear at small distances. The dispersion effects are connected with the deviation from quasi-neutrality, and with inertia of the electrons. By considering only one of these factors, one can obtain the steady motions which differ from a plane-parallel current. Two characteristic lengths correspond to them.

Let us investigate the manner in which the dispersion effects lead to the formation of steady waves for a cold plasma ($nmc^2 >> \frac{H^2}{8\pi} >> nT$). In investigating this case, we shall not consider the thermal motion, so that the system of equations - describing the motion of ions and electrons and the profile of the fields in the steady wave - assumes the form ($m_i = M$, $m_e = m$; the wave is propagated along the x-axis; the magnetic field is directed along the y-axis; Figure 4):

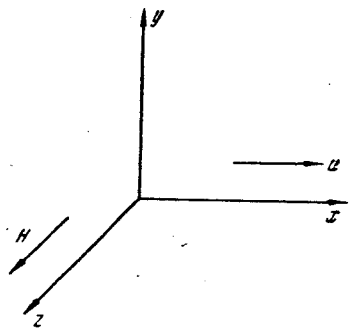


Figure 4

$$m_{i,e}\dot{v}_{x_{i,e}}\left(v_{x_{i,e}}-u\right)=\pm eE_x\pm\frac{e}{c}v_{y_{i,e}}H, \tag{27}$$

$$m_{i,e}\dot{v}_{y_{i,e}}\left(v_{x_{i,e}}-u\right)=\pm eE_y\mp\frac{e}{c}v_{x_{i,e}}H, \tag{28}$$

$$\left.\begin{aligned}E_y&=\frac{u}{c}(H-H_0),\\ -H'&=\frac{4\pi}{c}ne\left(v_{y,i}-v_{y,e}\right),\end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned}n_{i,e}&=\frac{n_0u}{u-v_{i,e}},\\ n_i&=n_e.\end{aligned}\right\} \tag{30}$$

The equation for the velocity components of electrons and ions along the x-axis is obtained from the last equation, which expresses quasi-neutrality. Excluding all variables, except H, from these equations, /43

we obtain (with accuracy up to terms with $\frac{m}{M}$)

$$-\frac{mc^2}{4\pi n_0 e^2 u}\frac{d}{dx}\left[\frac{dH}{dx}\left(\frac{H^2-H_0^2}{8\pi n_0 Mu}-u\right)\right]\left(\frac{H^2-H_0^2}{8\pi n_0 Mu}-u\right)=\left(\frac{H^2-H_0^2}{8\pi n_0 Mu}-u\right)H+uH_0. \quad (31)$$

This equation determines the form of the change in H in the steady wave being studied. Integrating once, we change it to the form

$$-a^2H'^2\left(\frac{H^2-H_0^2}{8\pi n_0 Mu}-u\right)^2=\frac{(H^2-H_0^2)^2-16\pi n_0 Mu^2(H-H_0)^2}{16\pi n_0 M}+C\quad\left(a^2=\frac{mc^2}{4\pi n_0 e^2}=\frac{c^2}{\omega_0^2}\right). \quad (32)$$

Considering the different values of C - the integration constant -, we obtain different solutions. It is convenient to trace the nature of the solution as a function of C, by plotting the integral curves in the phase plane (H, H'). These curves are shown in Figure 5.

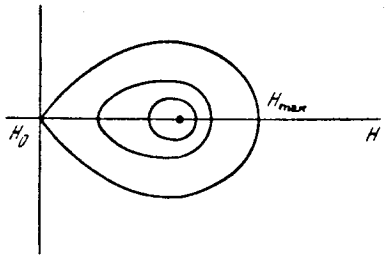


Figure 5

The solutions of equation (32) must describe periodic waves of finite amplitude. An exception to this is the solution corresponding to a special selection of the quantity C:

$$C=0.$$

This selection corresponds to the condition $\frac{dH}{dx}=0$ for $H=H_0$. Then the equation takes the form

$$\pm a\frac{dH}{dx}=\frac{(H-H_0)}{\frac{H^2-H_0^2}{8\pi n_0 Mu}-u}\cdot(16\pi n_0 M)^{-1/2}\cdot\sqrt{16\pi n_0 Mu^2-(H+H_0)^2}. \quad (33)$$

If a definite sign is selected before the root sign in equation (33), then it is impossible to construct a physically reasonable solution for H over the entire x-axis. However, continuous solutions exist everywhere (up to the second derivative, inclusively), in which

the derivative H' changes sign for certain $x = x_1$. At this point H reaches its maximum value H_{max}. The equation $\frac{dH}{dx}(x_1) = 0$ connects the amplitude of the magnetic field of H_{max} with the propagation velocity of the wave, and plays a role which is similar to the dispersion equation

$$16\pi n_0 M u^2 - (H_{max} + H_0)^2 = 0. \tag{34}$$

Solving equation (18) with respect to u, we obtain (Ref. 12 - Ref. 15) /44

$$u^2 = \frac{(H_{max} + H_0)^2}{16\pi n_0 M}. \tag{35}$$

In the limiting case of small amplitudes ($H_{max} \to H_0$), equation (35) determines the velocity of so-called magnetic sound. The propagation velocity of the wave increases with an amplitude increase. Integration of expression (33) gives the form of the change in H in such a wave. The form of the solution is symmetrical with respect to $x = x_1$, and represents a single impulse of the magnetic field with a width on the order of

$$\delta \sim \frac{c}{\omega_0},$$

where

$$\omega_0 = \left(\frac{4\pi n e^2}{m}\right)^{1/2}.$$

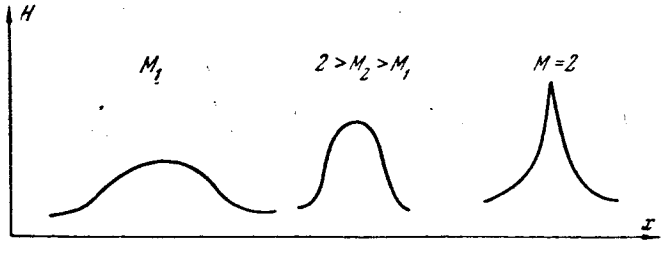


Figure 6.

Thus, the solution of equation (33) must describe the analog of an isolated wave in a rarefied plasma. A simple analytical expression for the form of the magnetic field in an isolated wave readily yields ($H_{max} - H_0 < H_0$) for small amplitudes. It has the form

$$H = H_0\left\{1 + 2\left(\frac{u^2}{\left(\frac{H_0}{\sqrt{4\pi n_0 M}}\right)^2} - 1\right)\operatorname{sh}^2\frac{x}{\frac{c}{\omega_0}\sqrt{\frac{u^2}{\left(\frac{H_0}{\sqrt{4\pi n_0 M}}\right)^2} - 1}}\right\}. \tag{36}$$

The dependencies $H = H(x)$ are given in Figure 6 for the different values of the Mach number $\mathcal{M} = \frac{u}{H_0/\sqrt{4\pi n_0 M}}$.

Equation (17) has an actual solution for u and H which are not arbitrarily large. Thus, for example, for isolated waves, solutions exist for $H_{max} \leq 3H_0$ (i.e., $u < 2\frac{H_0}{\sqrt{4\pi n_0 M}}$). As the amplitude of the wave approaches the critical amplitude, the density of the ions (electrons) on the crest of the wave strives to infinity. In physical terms, this means the following. An isolated wave represents a "hump" of the electric potential φ. In the coordinate system which is connected with the wave system, the stream of ions from $x = \infty$ runs against this potential barrier with velocity u. At amplitudes which are not too large, the initial critical energy of the ion $\frac{Mu^2}{2}$ exceeds the heighth of the potential barrier $e\varphi_{max}$ and the ions, somewhat delayed, pass through it. However, as follows from the solution, with an increase in the amplitude of the wave, the potential barrier becomesso much higher that $e\varphi_{max} > \frac{Mu^2}{2}$. The moment $e\varphi_{max} \approx \frac{Mu^2}{2}$ corresponds to the amplitude $H_{max} = 3H_0$ (in other words, the critical Mach number = 2). At the crest of such a wave, the ions, losing velocity, "are stopped", and their density increases up to infinity. For even larger amplitudes, the ions are simply "reflected" from the barrier, but the motion corresponding to such a picture has not yet been described within the framework of our original system of equations (27) - (30), since after reflection the flow becomes "multicurrent" (the interpenetrating streams of oncoming and reflected ions). /45

Thus, we can see that the dispersion effects cannot "reverse" the waves with a sufficiently large amplitude in a cold plasma.

If we take into account the thermal scatter of the ion velocities, then even at small wave amplitudes ions could be found which are reflected from the barrier (these are the ions with a low relative velocity $u - v_x$), i.e., which move initially in the propagation direction of the wave with a velocity which is close to u. These ions can be called "trapped". These ions withdraw energy from the wave, causing its damping. However, we can still disregard the damping. Without taking the trapped ions into consideration, we can readily find a solution for an isolated wave in a more general form by taking into account the thermal scatter, examining the distribution function of the ions with respect to velocity.

A very similar class of non-linear motions can exist in a plasma in the absence of an external magnetic field - these are non-linear ion-sound fluctuations. Since it is known from the linear theory that ionic sound only occurs for $T_e >> T_i$, we shall confine ourselves to this case.

Assuming that all of the quantities depend on x and t only in the combination $x - ut$, system (22) can be reduced to one differential

equation of the second order for the potential

$$\varphi'' = 4\pi n_0 e \cdot \left\{ \frac{u}{\sqrt{u^2 - \frac{2e\varphi}{M}}} - \exp\left(\frac{e\varphi}{T}\right) \right\}. \qquad (37)$$

Integrating equation (37) once, we obtain

$$-\frac{1}{2}(\varphi')^2 = 4\pi n_0 e \left(-\frac{uM}{e}\sqrt{u^2 - \frac{2e\varphi}{M}} - \frac{T}{e}\exp\frac{e\varphi}{T} \right) + C. \qquad (38)$$

Depending on the selection of the integration constant C, we can construct different periodic waves (see the integral curves on phase plane, Figure 7). The quantity C represents a special case; it is chosen from the condition $\varphi' \to 0$ for $\varphi \to 0$, i.e., $C = 4\pi n_0 (Mu^2 + T)$. This case can be especially distinguished on a phase plane. It corresponds to an isolated wave (Figure 8), which represents a symmetrical "hump" of the potential. /46

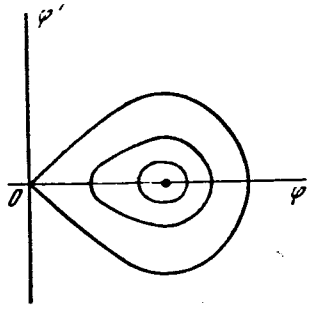


Figure 7.

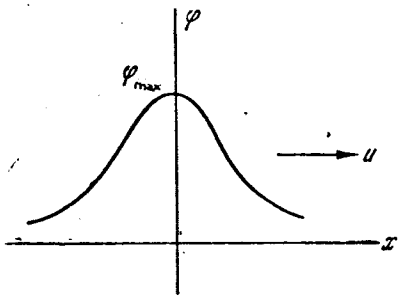


Figure 8.

The propagation velocity of such a wave u as a function of φ_{max} of the potential amplitude can be determined from expression (38), assuming that $\varphi' = 0$ for $\varphi = \varphi_{max}$ (Ref. 4),

$$u^2 = \frac{T}{2M} \cdot \frac{\left[\exp\left(\frac{e\varphi_{max}}{T}\right) - 1\right]^2}{\exp\left(\frac{e\varphi_{max}}{T}\right) - 1 - \frac{e\varphi_{max}}{T}}. \qquad (39)$$

In the limiting case of small amplitudes ($e\varphi_{max} << T$) u tends toward the velocity of isothermic sound $\sqrt{\frac{T}{M}}$. The profile of the potential in an isolated wave takes the following form for small amplitudes:

$$\varphi \approx \frac{3}{2}\frac{T}{e}\left(1 - \frac{T}{Mu^2}\right) \mathrm{sh}^2 \left\{ \frac{\sqrt{\pi n_0 e}}{\sqrt{T}} \sqrt{1 - \frac{T}{Mu^2}} \cdot x \right\}. \qquad (40)$$

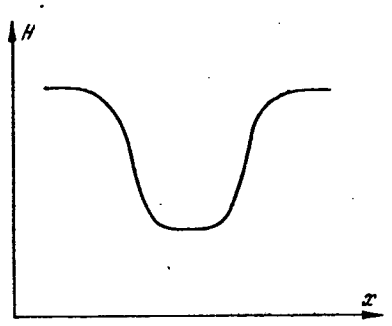
Ion waves also exist, but not at as large an amplitude as desired. Ultimately, the moment begins when the motion becomes

multiple current motion. This occurs due to the reason that the ions are not in a state to cross the potential barrier. The amplitude of an "isolated" wave, at which this occurs, equals $e\varphi_{max} = \frac{Mu^2}{2}$. If expression (39) is used for u - wave velocity -, then a transcendental equation is obtained for φ_{max}. The corresponding root is approximately equal to $e\varphi_{max} \approx 1.3T$. This is the critical amplitude of an isolated wave. The Mach number $\mathcal{M} = \frac{u}{\left(\frac{T}{M}\right)^{\frac{1}{2}}} \approx 1.6$ corresponds to it.

Thus, in both of the cases examined above (magneto-sound and ion waves), we obtained a similar picture of steady non-linear motion. In practical terms, the difference only consists of different spatial scales. This is not surprising, since the corresponding linear rules of dispersion for these cases [see equations (19) and (21)] are very similar. The most interesting are isolated waves. Isolated waves represent a special type of steady, non-linear waves. If periodic waves can exist for an arbitrary dispersion law (only not linear), then the definite nature of the dispersion law is required for isolated waves. This is related to the fact that, while spectral expansion of periodic waves contains the discrete ω and k, expansion of an isolated wave profile has a complex spectrum. Therefore, the reasoning employed at the beginning of this chapter is impossible. It is clear that for the existence of a stationary picture in an isolated wave, it is necessary that the smaller wave propagation velocities in the linear theory correspond to sections of the profile with larger velocity amplitudes. Then, the stronger influence of non-linearity upon the sections with greater velocity amplitude will be, roughly speaking, compensated by a decrease of $\frac{\partial\omega}{\partial k}$. The magneto-sound waves and "ion" sound in a plasma for $T_e \gg T_i$, which have been investigated by us, follow such a type of dispersion law ($\frac{\partial\omega}{\partial k}$ decreases with an increase in k). The dispersion law for waves on the surface of a heavy liquid in a canal of finite depth has such characteristics.

Waves with a reverse dispersion law, for which $\frac{\partial\omega}{\partial k}$ increases with an increase in k, must form isolated waves "of rarefaction" (Figure 9) in a non-linear case, in contrast to waves of "compaction", as in the preceding case. Such a dispersion law holds, for example, for waves across a magnetic field in a high-pressure plasma. It is not difficult to determine the profile of similar waves and to establish the relationship between the velocity of the wave and the amplitude of the magnetic field. However, we have already clarified the characteristic physical properties of non-linear, non-damped waves, and now we shall turn to a study of the different mechanisms for damping such waves.

3. Apart from the customary damping mechanisms - which are related



/47

Figure 9.

to a conversion of energy of ordered motion into heat during particle collisions - in a rarefied plasma there is "damping without collisions". It is related to "trapped" particles, i.e., to particles which have motion velocities which are close to the phase velocity of a wave. We shall study this phenomenon for the simplest example of electron Langmuir waves. Thus, the qualitative picture is no longer valid for any types. As is known, waves having a very small amplitude - which were investigated by L. D. Landau (Ref. 3) - damp out, if the distribution function of resonance frequencies decreases i.e., $\frac{df}{dv} < 0$ for $v = \frac{\omega}{k}$. /48

This is related to the fact that the more rapid frequencies are slowed down, and the slower ones are accelerated, by the wave. If in resonance there are fewer rapid frequencies than there are slow ones, then the wave is damped out. In practical terms, the linear theory rapidly becomes unusable, since in the damping process the form of $f(v)$ is changed. Thus, in a "quasi linear" theory, the distortion of the distribution function can be described by the equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v}, \tag{41}$$

where $D \sim E^2$ is particularly large for resonance frequencies.

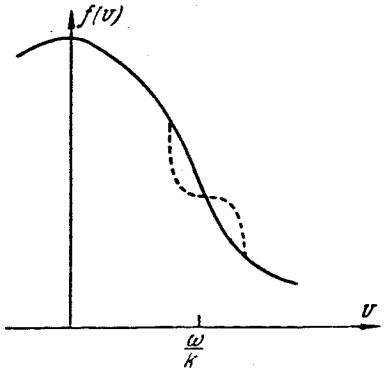


Figure 10.

According to equation (41), the resonance frequencies are redistributed, and a plateau $\frac{df}{dv} \to 0$ is formed in a diagram of the distribution function (Figure 10) [equation (41) is similar to the equation for thermal conductivity in a heterogeneous medium]. However, even though they may be infrequent, the collisions gradually "smooth out" the edges of the plateau and establish a quasi stationary state, in which $\frac{df}{dv}$ differs slightly from zero ($\frac{df}{dv} < 0$). It is necessary to introduce a collision term in equation (41) in order to determine its magnitude and the damping. It is clear that the greater the fluctuation amplitude, the stronger their relaxing effect upon the form of the distribution function in the vicinity of $\left(v \approx \frac{\omega}{k}\right)$. Consequently, it can be expected that the magnitude of the damping decrement $\frac{1}{\varepsilon}\frac{d\varepsilon}{dt}$ (ε is the wave energy) - which is proportional to $\frac{df}{dv}\ v = \frac{\omega}{k}$ - will decrease with an increase in ε . We can determine the stationary inclination $\frac{df}{dv}$ from the equation /49

$$\frac{d}{dv} D \frac{df}{dv} = \mathrm{Sh}(f), \tag{42}$$

where the expression for the quasi linear diffusion coefficient D can be simplified in the following way for resonance frequencies $\left(v \approx \frac{\omega}{k}\right)$:

$$D(v) = \sum_k \frac{e^2}{m^2} E_k^2 \,\mathrm{Im}\,(\omega + kv)^{-1} \sim \frac{e^2 E^2}{m^2 \omega}. \tag{43}$$

In the expression for $\mathrm{St}(f)$, we retain the chief term containing the second derivative

$$\mathrm{St}(f) \approx \nu_{\text{chief}} \frac{T}{m} \frac{d^2 (f_0 - f)}{dv^2},$$

where f_0 is the Maxwell distribution function. This simplified form of the "collision integral" takes into account the reduction of the local equilibrium distribution. Integrating equation (42) once, we obtain

$$\frac{df}{dv} = \frac{df_0}{dv} \cdot \frac{1}{1 + \frac{e^2 E^2}{m\omega T \nu_{\text{chief}}}}. \tag{44}$$

It can be seen from this expression that for waves having a small amplitude $\frac{e^2 E^2}{m\omega T \nu_{\text{chief}}} < 1$ the damping decrement

$$\nu = \frac{\pi\omega_0}{2} \cdot \left(\frac{\omega}{k}\right)^2 \frac{df}{dv}\left(v = \frac{\omega}{k}\right) \to \frac{\pi\omega_0}{2} \cdot \left(\frac{\omega}{k}\right)^2 \frac{df}{dv}\left(v = \frac{\omega}{k}\right) = \nu_0$$

tends toward the ν_0-decrement of the Landau damping. However, at

amplitudes of $\frac{e^2E^2}{m\omega T\nu_{chief}} > 1$, the linear theory is not applicable. As follows from expression (44), the damping decrement for such waves must decrease with an increase in amplitude, as E^{-2}.

As was shown above, this is valid only for the damping of fluctuations which represent sufficiently wide wave packets, since we have to make use of quasi linear theories. For a monochromatic wave with one ω and k, expression (44) is still not applicable, and it would be necessary to conduct a special investigation. We are confining ourselves here to only semi-quantitative estimates in order to establish the dependence of the increment upon amplitude. Formula (44) can be interpreted in the following way: let us present it in the form $\nu = \nu_0/(1 + \tau_1/\tau_2)$. Here ν_0 is the decrement obtained in the linear approximation (Landau damping); τ_1 is the characteristic time for establishing the local Maxwell distribution; τ_2 is the characteristic time for the distortion of the distribution function under the effect of the wave packet field. If $\tau_1 \ll \tau_2$ (due to the collision, the Maxwell distribution function is established), we obtain the normal Landau damping. With an increase in the wave amplitude, the distortion which is introduced by it becomes so large that the collisions can not establish the Maxwell distribution function, and the damping decrement decreases. It is possible to estimate the absorption for a monochromatic wave by means of a similar interpretation, if τ_1 and τ_2 are chosen correctly. Let the potential amplitude in the wave be φ; then the frequencies with velocities on the order of $\pm\sqrt{\frac{e\varphi}{m}}$ with respect to the wave will account for the absorption. /50

This means that the distribution function is distorted to the greatest extent in the region Δv, with a width on the order of $\pm\sqrt{\frac{e\varphi}{m}}$. Due to Coulomb collisions with scattering at small angles, the local equilibrium in this will be apparently established for time $\tau_1 \sim \frac{e\varphi}{\nu_{chief}T}$. The time for the non-linear distortion under the effect of the wave field is on the order of $\tau_2 \sim \frac{\lambda}{\sqrt{e\varphi m}}$, where λ is the wavelength. Finally, we obtain (Ref. 4)

$$\nu = \frac{\nu_0}{1 + \frac{(e\varphi)^{3/2}}{T\lambda\nu_{chief}\sqrt{m}}}. \qquad (45)$$

This means that for a monochromatic wave with an increase in the amplitude, the damping decrement decreases as $E^{-\frac{3}{2}}$. A strict investigation confirms this dependence (Ref. 16).

4. The conclusion can be drawn from these results that non-linear waves in a rarefied plasma damp out very slowly, as only "relaxation" occurs in the distribution of resonance frequencies which

account for the damping. Nevertheless, this does not guarantee the fact that, once formed, the non-linear steady waves can exist for a long time. It is still necessary to determine whether they are stable with respect to different random distortions. If they prove to be unstable, then this will mean that their energy is converted into some other types of plasma motion - possibly into non-ordered turbulent motion. Then one could speak of effective damping. It is clear that the extension of rather large disturbances in a rarefied plasma must be accompanied by significant deviation of the plasma from thermodynamic equilibrium, which can lead to instability.

For example, let us examine a steady, isolated wave which is propagated across the magnetic field in a cold plasma $\left(nT << \frac{H^2}{8\pi}\right)$. Let us again turn to the picture of the motion of ions and electrons of the plasma in such a wave. If $\frac{H^2}{8\pi} << nmc^2$, the plasma is quasi neutral. The ions and electrons move at the same velocities in the direction of the wave propagation. However, the electric current in /51
a direction which is perpendicular to the wave velocity and to the magnetic field is formed only by electrons. In a homogeneous plasma, as is known, the presence of significant, relative motion of ions and electrons leads to the so-called "bunched" instability. It is clear that an analogous effect can be expected here. The problem can be simplified, if - in investigating the small deviations from the stationary picture of an isolated wave - the terms which take into account the non-disturbed motion of a plasma in the x-direction are disregarded. It is apparent that this is valid in the case when the time required for the development of instability is considerably less than the time for the plasma "to pass" the region of the isolated wave.

With respect to the order of magnitude, the latter is apparently equal to $\frac{\delta\sqrt{4\pi nM}}{H}$, where δ is the "width" of the wave. This problem is not difficult to solve, if - for purposes of simplicity - the disturbed motion of ions and electrons is regarded as the motion of two "liquids" with adiabatic laws of pressure change. We shall not take into account the effect of the magnetic field on the disturbed motion, but shall limit ourselves to the frequency fluctuations which are considerably larger than the Larmor electron frequency ω_{H_e}. In this approximation, the equations for the disturbed quantities v_e (electron velocity); v_i (ion velocity); n_e, n_i (ion and electron densities) and φ (electric potential) assume the form:

$$\left.\begin{aligned} i(\omega + kv_0)\,\mathbf{v} &= \nabla\frac{e}{m}\varphi - \nabla\frac{T}{mn_0}n; \\ i\omega\mathbf{V} &= -\nabla\frac{e}{M}\varphi; \end{aligned}\right\} \qquad (46)$$

$$\left.\begin{aligned} i(\omega + kv_0)\, n_e + ikn_0 v_y + \frac{d}{dx}(n_0 v_x) = 0; \\ i\omega n_i + \frac{d}{dx}(n_0 V_x) + ikn_0 V_y = 0; \\ -k^2\varphi + \varphi'' = 4\pi e(n_e - n_i). \end{aligned}\right\} \qquad (47)$$

Here, equations (46) represent equations of motion for electrons and ions, and (47) represents equations of continuity for electrons and ions, and also the Poisson equation for the electric field. We chose the dependence of the disturbed quantities in the form $\varphi(x)\, e^{i(\omega t + ky)}$. The terms v_0, T and n_0, which enter into the equations, represent the undisturbed mean electron velocity (along the y axis), the electron temperature (for purposes of simplicity, we shall assume the ions are cold) and the plasma density - all of which are dependent on x. On the assumption that the derivatives with respect to x of the disturbed quantities are considerably greater than the derivatives of the undisturbed ones ("a quasi classical" approximation) this system of equations can be reduced to one differential equation of the second order for the quantities n_e: /52

$$\frac{T}{m}\frac{d^2 n_e}{dx^2} + \left[(\omega + kv_0)^2 - \frac{T}{m}k^2 - \frac{\omega_0^2}{1 - \frac{\Omega_0^2}{\omega^2}}\right] n_e = 0. \qquad (48)$$

A study of the stability can be reduced to the problem of eigen values for the equation (48). With the aid of the boundary conditions, solutions are selected which decrease at both sides of the isolated wave.

Let us study the behavior of the function

$$F(x, \omega, k) = (\omega + k v_0)^2 - \frac{T}{m}k^2 - \frac{\omega_0^2}{1 - \frac{\Omega_0^2}{\omega^2}}.$$

In a homogeneous plasma, this function would not depend on x, and the dispersion equation connecting ω and k would have the form

$$F(\omega, k) = 0. \qquad (49)$$

This equation yields unstable solutions, when $v_2^0 > \frac{T}{m}$, i.e., when the mean velocity of the relative motion of the ions and electrons exceeds the thermal velocity of the electrons. For $k\left(k^2 << \frac{\omega_0^2}{T}\, m\right)$ which are not very large, this equation can approximately be written as

$$F(\omega, k) \approx k^2\left(v_0^2 - \frac{T}{m}\right) - \frac{\omega_0^2}{1 - \frac{\Omega_0^2}{\omega^2}} = 0.$$

We thus find

$$\omega^2 = \frac{\Omega_0^2 k^2 \left(\frac{T}{m} - v_0^2\right)}{\omega_0^2 - k^2\left(v_0^2 - \frac{T}{m}\right)}.$$

For $v_0^2 > \frac{T}{m}$, ω becomes imaginary (instability). Turning to our homogeneous problem, let us examine the spatial behavior of the function $F(x, \omega, k) \approx k^2\left(v_0^2 - \frac{T}{m}\right) - \frac{\omega_0^2}{1-\Omega_0^2/\omega}$ (in this approximation, it is sufficient to examine the actual ω^2). For purposes of visualization, let us represent (Figure 11) the profile of the change v_0 and ω_0^2 in an isolated wave as a function of x. In the region where $v_0^2 > \frac{T}{m}$, i.e., $F(x, \omega, k) > 0$, for n_e there is an oscillating solution. This is far from an isolated wave $F(x, \omega, k) < 0$ which corresponds exponentially to a damped solution. At the "reversal" point, where $F(x, \omega, k) = 0$, these solutions "join". Thus, the necessary local solutions always exist, and instability appears in that case if there is a region where $v_0^2 > \frac{T}{m}$ within an isolated wave. As is known, the increments for the increase of such instability are on the order of Ω_0 (in a plasma of "zero" temperature, the maximum increment is greater than $\Omega_0\left(\frac{M}{m}\right)^{1/6}$). Several simplifying assumptions were made in obtaining these results. In the general case, the problem is somewhat complicated: the equation corresponding to equation (48) becomes an equation of the fourth order; the points at which the "joining" occur are mixed in the complex plane x. However, the condition of instability $v_0^2 > \frac{T}{m}$ remains unchanged.

/53

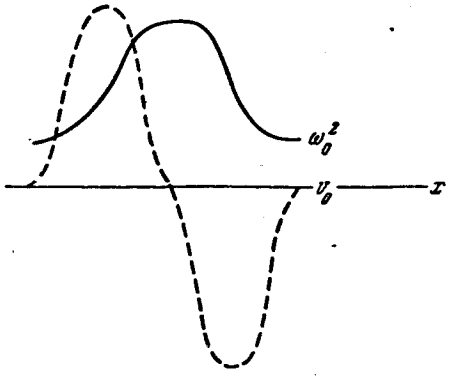


Figure 11.

The amplitude of v_0 in an isolated wave is increased with an increase in the Mach number $\mathcal{M}\left(\mathcal{M} = \frac{u\sqrt{4\pi n_0 M}}{H_0}\right)$. Finally, for a certain value of the Mach number ($\mathcal{M} = \mathcal{M}^*$), the amplitude of v_0 exceeds the mean thermal velocity of the electrons, i.e., the wave becomes unstable. If the solution of the problem regarding the profile of an isolated wave, which was set forth previously, is used, it can be determined that for a cold plasma $\left(nT << \frac{H_2}{8\pi}\right)$

$$\mathcal{M}^* \approx 1 + \frac{3}{8}\left(\frac{8\pi n_0 T}{H_0^2}\right)^{1/3}. \qquad (50)$$

In essence, all of this means that for heterogeous problems it is possible - by taking known precautions - to apply the the criterion of "bunching" of instability in a homogeneous plasma. Let us now determine whether the original assumption is satisfied regarding the fact that the time required for the development of instability is considerably less than the time for the plasma "to pass" the region of an isolated wave $\frac{c}{\omega_0}\cdot\left(\frac{H_0}{\sqrt{4\pi n_0 M}}\right)^{-1} >> \nu^{-1}$. Setting $\nu \sim \Omega_0$, we obtain $H_0^2 << 4\pi n_0 mc^2$.

The development of bunched instability, which influences the original non-linear wave, must cause an effective damping of the latter - the energy of ordered motion of the electrons in a non-linear wave changes into energy of non-ordered Langmuir electron fluctuations. In this sense, the effect of such instability can be treated as the friction force of electrons on ions, which has a collective nature (Ref. 4, Ref. 17, Ref. 18).

Although we have applied the instability being considered to only isolated steady waves, it is clear that such a situation could arise for other waves in a plasma with a magnetic field. Nevertheless, "bunched" instability is related to a fairly restricted class of problems. The so-called "decay" instability, which can be observed in periodic non-linear waves (Ref. 19), has a more general nature. /54

Let us begin with certain general remarks. In a study of the stability of non-linear stationary waves (which we shall designate as "background" for purposes of brevity), it is convenient to change to a coordinate system which moves with a wave. The coefficients in linearized equations, which describe the conduct of small deviations from the "background", do not depend on time, and the dependence of the solutions for these equations upon time can be expressed in the

form $e^{i\omega t}$. The problem can be reduced to the solution of a system of equations which can be written symbolically in the form

$$\hat{L}\varphi = 0, \tag{51}$$

where $\hat{L}$ is a certain linear differential operator. The explicit form of the operator depends on the "background" and the eigen frequency ω , the determination of which is included in a study of stability. The operator $\hat{L}$ can be represented in the form of the sum $\hat{L}_0$ and $\hat{L}_1$, where $\hat{L}_0$ is the differential operator with constant coefficients, and $\hat{L}_1$ is the differential operator which strives to zero, along with the striving to zero of the amplitude being studied for stability of a stationary wave. For waves with a finite, but small amplitude, $\hat{L}_1$ is small, and it is natural to use the theory of disturbances. In the zero approximation[1] equation

$$\hat{L}_0\varphi = 0 \tag{52}$$

describes the fluctuations of a homogeneous plasma with eigen functions which are proportional to e^{ikr}, and eigen values ω which satisfy the dispersion equation $\omega = \omega(k)$. In the first approximation of $\hat{L}_1$, diagonal matrix elements $\langle \varphi_\omega | \hat{L}_1 | \varphi_\omega \rangle$ appear, while the spatial dependence of $\hat{L}_1$ is determined by the multipliers $e^{\pm i, k_0, r}$. It is clear that the matrix elements change to zero, if only one value of the wave vector modulus k corresponds to each value of the frequency ω . The first approximation of the theory of disturbances provides a non-vanishing contribution only if "degenerate" states exist for which at least two wave vectors (k_1 and k_2) correspond to one ω . In this case, the following relationship must be fulfilled between k_1 and k_2: /55

$$k_1 = k_0 + k_2, \tag{53}$$

The fact that they belong to one and the same frequency can be written in the form $\omega_1 = \omega_2$. If we now turn from a system of coordinates moving with a wave to a laboratory system of coordinates, the frequencies ω_1 and ω_2 will be different. Thus, the following condition will be fulfilled

$$\Omega_1 = \Omega_0 + \Omega_2, \tag{54}$$

where Ω_0 is the frequency of the background fluctuations ($\Omega_0 = k_0 u$) and Ω_1 and Ω_2 represent the frequencies corresponding to the wave vectors k_1 and k_2 ($\Omega_1 = \omega_1 + k_1 u$; $\Omega_2 = \omega_2 + k_2 u$). Conditions (53) and (54) can be regarded as laws of conservation of quasi energy and quasi impulse in the interaction ("decay") of quasi frequencies of the waves. Therefore, from this point on we shall call them decay conditions,

[1] If dissipation is disregarded, in a hydrodynamic approximation $\hat{L}_0$ is a self-adjoint operator, and its eigen functions must correspond to non-damped waves.

and the instability which thus arises we shall call the decay of a fluctuation with the frequency Ω_0 and wave vector k_0 into a fluctuation with the frequencies Ω_1 and Ω_2 and wave vectors k_1 and k_2. The decay conditions are not fulfilled for every dispersion law $\omega(k)$. The curves corresponding to different forms of the spectra are shown in Figure 12.

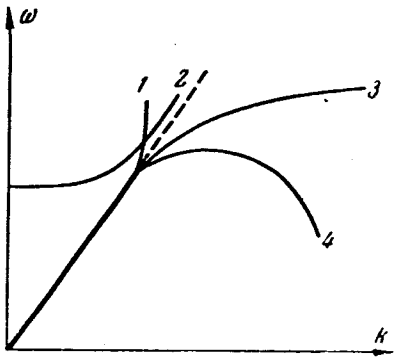


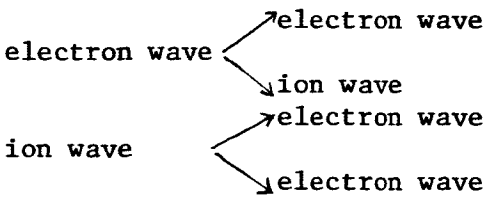
Figure 12.

It can be readily shown that the fluctuation decays can only occur for spectra 1 and 4. Fluctuations having spectra which are similar to spectra 2 and 3 are stable with respect to decay. However, in the presence of several branches in the fluctuation spectrum, fluctuations which characterize the spectra which are analogous to spectra 2 can be unstable with respect to decay into fluctuations, of which only one does not belong to a given branch. More precisely: decays are possible when it is possible to draw a curve which is similar either to curve 1 or to curve 3 (for different spectra branches, however, "prohibitions" connected with wave polarization can arise) through three points which correspond to the fluctuations Ω_0, Ω_1, Ω_2 (generally speaking, these three points can lie at different branches). But the fulfillment of "decay" conditions by itself doe not indicate instability. If a correction is made for the frequency from the first approximation of the theory of disturbances, it appears that it is either imaginary - i.e., instability actually exists - or it is real - i.e., only a shift of the frequency occurs. What actually occurs must be studied in each explicit case. The value of A_i, characterizing the "background" being investigated, can be written in the form /56

$$A_i = A_{i0} + 2\delta(k_0)\sin k_0 r + O(\delta A_i^2),$$

where k_0 is the wave vector. From this point on we shall not take the component $O(\delta A_i^2)$ into consideration - in other words, we shall study the stability of the main harmonics, considering its interaction with small deviations from the background. For this purpose, we shall first determine in what cases the decay conditions can be fulfilled.

Let us first examine the simplest case - a plasma without a magnetic field. In such a plasma there are two fluctuation branches: electron longitudinal fluctuations and ion sound (for $T_e \gg T_i$). The electron fluctuations belong to a mode, whose spectrum corresponds to curve 2 (see Figure 12), and the ion fluctuations - to a mode whose spectrum corresponds to curve 3. Consequently, these fluctuations are stable in themselves. However, cross-over decays are not prohibited:



The "decay" of an electron longitudinal wave into electron and ion longitudinal waves is one of the simplest examples of instability of the "decay" type. Equations for small disturbances take the form:

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)v_i-\frac{e}{M}E=-2\delta v_i\frac{\partial}{\partial x}(v_i\sin k_0x);$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)n_i+n_0\frac{\partial v_i}{\partial x}=-2\frac{\partial}{\partial x}\{(n_i\delta v_i+$$

$$+v_i\delta n_i)\sin k_0x\};$$

$$eE+\frac{T}{n_0}\frac{\partial n_e}{\partial x}=2\frac{T}{n_0^2}\cdot\frac{\partial}{\partial x}(n_e\delta n_e\sin k_0x) \qquad (55)$$

(this system of equations will describe an ion wave). Here v_i, $n_{i,e}$ represents the velocity and density of the ions (electrons).

For an electron wave, the hydrodynamic equations take the form:

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)v_e+\frac{e}{m}E+\frac{1}{mn_0}\frac{\partial p_e}{\partial x}=-2\delta v_e\frac{\partial}{\partial x}(v_e\sin k_0x)+$$

$$+\frac{\gamma p_0}{mn_0^3}\cdot 2\delta n_e\frac{\partial}{\partial x}(n_e\sin k_0x);$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)n_e+n_0\frac{\partial v_0}{\partial x}=-2\frac{\partial}{\partial x}\{(n_e\delta v_e+v_e\delta n_e)\sin k_0x\}; \qquad (56)$$

/57

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\left(p_e-\gamma\frac{p_0}{n_0}n_e\right)=2\gamma(\gamma-1)\frac{p_0\delta n_e}{n_0^2}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\times$$

$$\times(n_e\sin k_0x);$$

$$\frac{\partial E}{\partial x}=-4\pi en_e,$$

where v_e is the electron velocity, γ is the adiabatic exponent for electrons, which can be set equal to 3 in a one-dimensional case; p_e is the electron pressure; δv, δn are the amplitudes of velocity and density of electrons and ions in the original electron wave being studied for stability. For purposes of simplicity, we are confining ourselves here to the one-dimensional case, i.e., we assume that the quantities describing the dynamics of small deviations from the "background" depend only on time and the x coordinates (the x-axis is directed along k_0).

Thus, in accordance with the indicated scheme for studying stability, let us look for the disturbance in the form of the superposition $C_i e^{i(\omega + k_1 x)} + C_e e^{i(\omega + k_1 x)}$ of an ion and electron wave. In the zero approximation, they are independent, but a relationship appears between them if the right parts of the equations (55) and (56) are taken into account. From the condition of solvability of a system of equations for C_i and C_e, it is possible to obtain the following expression for the square of the imaginary part of the frequency after simple, but rather rough, calculations:

$$\nu^2 \approx \left(\frac{\delta v_e}{u}\right)^2 \frac{k_2 u}{4} \frac{\Omega_1}{\Omega_2} \left\{\Omega_2 + (\gamma - 2)\, k_2 v_T \frac{v_T}{u}\right\}, \tag{57}$$

where $v_T = \sqrt{\frac{\gamma p_0}{n_0 m}}$ and Ω_1, Ω_2 represent the frequencies of ion and electron fluctuations in a laboratory system of coordinates.

The decay conditions take the following form:

$$\begin{aligned} \pm |k_1| &= k_0 \pm |k_2|, \\ \pm k_1 u_i &= \sqrt{\omega_0^2 + k_2^2 v_T^2} - \sqrt{\omega_0^2 + k_0^2 v_T^2}, \end{aligned} \tag{58}$$

where $u_i = \sqrt{\frac{p_0}{n_0 M}}$.

Utilizing expressions (57) and (58), we can show that $\nu^2 > 0$, i.e. electron longitudinal fluctuations are unstable with respect to decay into ion and electron longitudinal fluctuations. Shortwave fluctuations are the most unstable ($k_0 \lesssim \varkappa$; $\frac{1}{\varkappa}$ - Debye radius). For them

$$\nu_{max} \sim \frac{\delta v}{u} \sqrt[4]{\frac{m}{M}} \cdot \omega_0. \tag{59}$$

All of the calculations performed above were carried out in the hydrodynamic approximation, i.e., the thermal motion of the electrons was taken into account only by including the electron pressure. As is known,

consideration of thermal motion also leads to the phenomenon of wave damping. Damping of electron waves cannot be considered, if the wave vector $k << \varkappa$. For ion waves, the damping is not an exponentially small effect, and therefore the instability of the background will be determined, in fact, by the inequality $\nu > \nu_i$, where ν_i is the damping decrement of the ion fluctuations. As is known [see, for example, (Ref. 20)], for $p_i << p_e$ /58

$$\nu_i \cong \sqrt{\frac{\pi}{8}}\,\Omega_1 \sqrt{\frac{m}{M}}. \tag{60}$$

Comparing this expression, for example, with formula (59), we find that fluctuations for which the amplitudes satisfy the inequality $\frac{\delta v}{u} > \left(\frac{m}{M}\right)^{3/4}$, will be unstable. As can be seen from expressions (57) and (58), we should note that electron fluctuations with frequencies which are less than the frequencies of the background are formed during the decay.

Studying the second case of cross-over decay in a similar way, we can verify the fact that ion waves are stable: the relationship between C_1 and C_2 leads only to a shift in frequency. If we proceed in the same way as in the first case, it will be possible to study the stability of a different type of non-linear periodic waves having an amplitude which is not too large, and in a plasma located in a magnetic field. Such a study was carried out in one of the works (Ref. 21) with Alfvén magnetohydrodynamic waves . It is well-known that in magnetic hydrodynamics (and, in addition, not only in an incompressible liquid, but also in a gas) Alfvén waves represent an exact solution of non-linear equations. Therefore, the assumption can be made that these waves exist indefinitely without a change in form. Analysis has shown, however, that an Alfvén wave "decays" into the sum of two waves: an Alfvén and a slow magneto-sound wave (or a rapid magneto-sound and a slow magneto-sound wave). The increments of "decay" in stability are proportional to the first power of the amplitude of the original non-linear wave. Therefore, waves with a small amplitude can exist for a long period of time, and not decay.

A more accurate study of the equations $\hat{L}_0\varphi = \hat{L}_1\varphi$, which were obtained from an analysis of decay instability, shows that - on the basis of the form of the dispersion law - a guess can be made as to whether there will be a correction for the frequency which is imaginary (instability) or real (shift of the frequency). If $|\Omega_0| > |\Omega_1|, |\Omega_2|$, then in the fulfillment of the decay conditions (53) and (54) the original wave with a frequency Ω_0 is unstable. As will be seen below, "decay" instability plays an important role in the theory of non-collision shock waves, which we shall now discuss.

3. SHOCK WAVES IN A STRONGLY RAREFIED PLASMA

In the voluminous amount of literature on this subject, which has accumulated during the last several years, completely different - and, it would seem, even contradictory - statements are encountered. In the /59
first approximation, it is possible to divide them into two groups of conflicting points of view:

1) Shock waves with a thickness which is considerably less than the mean free path exist, while everything occurring within the front, in principle, can be described within the framework of laminar theory (in terms of ordered, non-linear fluctuations);

2) A process of anomalous dissipation in the front of the shock wave is connected with plasma turbulence.

Finally, there is a third, negative point of view, which states that such shock waves do not exist, generally speaking. There are weak points in the arguments given by the supporters of the different theories, which prevent one from making an unequivocal choice of one of the theories. Thus, in the turbulence theory, the mechanism for the instability leading to a change into a turbulent state is not clear. Within the laminar theory itself there is no unity - frequently, it would seem, contradictory results are obtained.

The natural course to be followed in constructing a theory of shock waves without collisions must be as follows. First a laminar theory is constructed, which is based on the concept of regular fluctuations (in order to do this, it seems satisfactory for us to take the information set forth in the preceding section as a basis), and then the stability of such solutions is studied. Finally, in unstable cases (and also when, in general, laminar solutions do not exist) a "turbulent" theory is constructed.

1. A laminar theory can be quite readily constructed. It is sufficient to take into consideration only the effect of damping upon the nature of non-linear steady waves. In the absence of damping, these waves describe reverse motions. Thus, the state of the plasma up to and after the passage of an isolated wave would appear to be the same. It is clear that consideration of dissipation must disturb reversibility, and the states of the plasma up to and after an isolated wave will be different. If equations of conservation of currents of mass, impulse, and energy are used for non-linear motions of the plasma, then for steady motions these equations must connect the states conforming to the equations of the Hugoniot adiabatic curve, by definition. If the damping is not taken into consideration, the states up to and after an isolated wave trivally satisfy the Hugoniot adiabatic

curve. In what way does the form of an isolated wave change, if dissipation is taken into consideration? The state after the passage of an isolated wave must differ from the original state and this difference, naturally, is determined by the explicit mechanism and magnitude of the dissipation.

On the other hand, the Hugoniot adiabatic curve does not depend on dissipation. In normal gas dynamics, in the theory of the thickness of a shock wave this seeming paradox is resolved by the fact that the form itself of the transitional zone (its thickness) changes as a function of the coefficients of viscosity, thermal conductivity, and other quantities which characterize dissipation. In a rarefied plasma, the "thickness" of an isolated wave (for small dissipation) is set independently of the /60
Hugoniot adiabatic curve by the dispersion properties. The solution of the seeming paradox in the case of a rarefied plasma can be found in the fact that - after the passage of an isolated wave - the state of the plasma remains "disturbed". Intense fluctuations remain in the plasma, whose contribution to the currents of impulse and energy must be taken into account. This means that within the front of the shock wave regular fluctuations of a finite amplitude spontaneously accumulate. As is known, the thickness of the front of a weak shock wave in normal gases considerably exceeds the mean free path. In a study of the structure of the front, this fact makes it possible to use the equations of gas dynamics with consideration of the dissipative effects.

Let us begin our investigation with the shock waves in a plasma in a magnetic field. In the case of a rarefied plasma located in a magnetic field, when the mean free path is considerably greater than the mean Larmor radius of the ions, a formal gas-dynamics description is used (for motions across the line of force) within spatial regions which are less than the free path. It is only necessary that all the quantities change very little at a distance on the order of the Larmor radius. In investigating the structure of the front of a shock wave which is propagated across the magnetic field in a rarefied plasma, we shall assume that everywhere within the front the condition of the smallness of the Larmor radius - as compared with any characteristic dimension - is fulfilled. For this purpose, it is necessary that the condition of the smallness of the wave amplitude be fulfilled. Let us first examine the simpler case of a "cold" plasma $\left(p \ll \frac{H^2}{8\pi}\right)$. First, let us conceive of the Joule heating due to collisions between ions and electrons as the damping mechanism (as we shall see below, the concrete damping quantity will have a purely symbolic nature in the given problem). Our problem consists of establishing a system of differential equations for the quantities characterizing the plasma and self-consistent electromagnetic fields within the front of a shock wave, and of studying it. Let us introduce

a system of coordinates, in which the front of the wave is at rest; we shall direct the magnetic field along the z-axis. Let zy be the plane of the front. The electric current will be carried by electrons in the y direction (Figure 13). The inertia of the electrons in this direction will have a significant effect upon the structure of the front. Finally, for purposes of simplicity, we shall assume that within the front the condition of quasi neutrality $n_i = n_e$, is fulfilled, where /61 n_i, e is the density of a number of ions (electrons) in accordance with the analogous problem regarding non-damped non-linear fluctuations across the magnetic field (see Section 2).

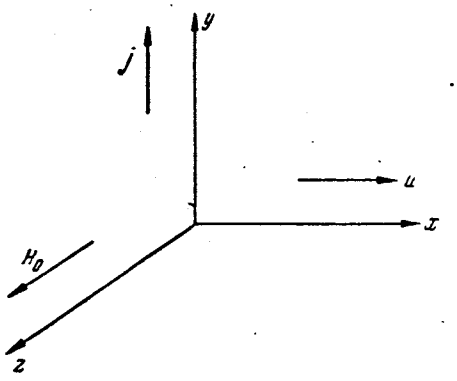


Figure 13.

The following are included in the group of quantities determining the plasma and the field: n, H, v - the velocity of the plasma in the direction of the wave propagation, v_y - velocity of electrons carrying the current; E_y - charge of the electric field along the y-axis (the field along the E_x-axis does not enter into the equations due to quasi neutrality). For these six unknowns there are six equations: a) the equation for conservation of the stream of particles; b) equation of conservation for the impulse stream; c) equation for the conservation of the energy stream; d) the equation of motion of electrons in the direction of the transfer of electric current - along the y-axis; e) and f) Maxwell equations for the corresponding components of the curls E and H. The original system - of six equations - after simple transformations can be reduced to a differential equation of the second order for one of these variables, for example, H. However, since the entire scheme of gas-dynamics approximation holds only for weak shock waves, it is possible to simplify the equations from the very beginning. Actually, in a weak wave propagated in a cold plasma, the pressure drop of the plasma will be negligible as compared with the drop in the magnetic field: $\frac{p}{H_0^2} << \frac{H - H_0}{H_0}$. Then, the velocity of the plasma v can be directly expressed by H from the equation for conservation of an impulse stream. There is no necessity for us to use an equation for the conservation of an energy stream now, since p clearly does not enter into the remaining equations (the original system of equations is

split). Let us write these equations, taking into account the given approximations:

$$\left.\begin{aligned} \frac{d}{dx} nv &= 0; \\ \frac{d}{dx}\left(\frac{Mnv^2}{2} + \frac{H^2}{8\pi}\right) &= 0; \\ mnv\frac{dv_y}{dx} &= -enE_y + \frac{e}{c} nvH - \bar{\nu} mnv_y; \\ \frac{dE_y}{dx} &= 0; \\ \frac{dH}{dx} &= \frac{4\pi ne}{e} v_y. \end{aligned}\right\} \qquad (61)$$

The last term in the right part of the equation of motion for electrons corresponds to the friction force of an electron gas on an ion gas ($\bar{\nu}$ is the mean frequency of collisions between an electron and ion, equal to $n < v_e\sigma >$, where n is the density of a number of ions; σ is the collision cross-section, and v_e is the relative electron-ion velocity; in a slightly weak shock wave $\bar{\nu}$ can be regarded as approxi- /62 mately constant within the front); M, m is the mass of an ion (electron).

After all the variables, except H, have been excluded, the system of equations (61) can be reduced to the following differential equation for determining H :

$$-a^2\frac{d^2H}{dx^2} = H_0 - H + H\frac{H^2 - H_0^2}{8\pi Mn_0u^2} + \frac{a^2}{u}\nu\frac{dH}{dx}. \qquad (62)$$

Here H_0 is the magnetic field in a plasma up to the arrival of the shock wave (for $x \to \infty$); n_0 is the understood density of a number of ions (electrons); u is the velocity of a shock wave with respect to an undisturbed plasma;

$$a^2 = \frac{mc^2}{4\pi ne^2} = \frac{c^2}{\omega_0^2}.$$

If friction is excluded from this equation, it will be similar to equation (31). The only difference consists of the fact that here we have confined ourselves to small amplitudes, for purposes of simplicity. Equation (2) represents an equation of motion of an anharmonic oscillator in the presence of friction; H plays the role of a generalized coordinate; x -the role of time.

The form of the hole is determined by the potential

$$V(H) = \frac{1}{2}(H - H_0)^2\left[\frac{(H + H_0)^2}{16\pi n_0 Mu^2} - 1\right]. \qquad (63)$$

For purposes of visualization, the form of $V(H)$ is shown in Figure 14.

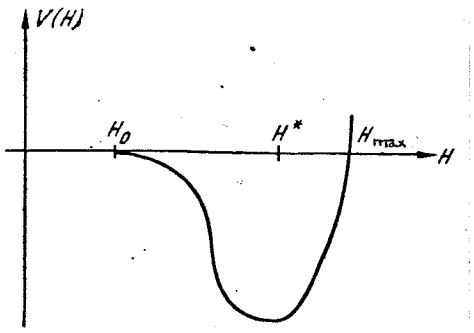


Figure 14.

For

$$H = H^* = -\frac{H_0}{2} + \sqrt{8\pi n_0 M u^2 + \frac{H_0^2}{4}}$$

$V(H)$ reaches a minimum. An analogy with the oscillator makes it possible to establish the profile of H readily within the front of the shock wave. H oscillates around the values H^* with damped amplitude up until the time when $H = H^*$ is established, which corresponds to the magnetic field behind the front of the shock wave. In order that H_0 correspond to the minimum of the magnetic field in the wave, i.e., in order that $V(H)$ have the form which is shown in Figure 14, it is necessary that the condition $u^2 > \frac{H_0^2}{4\pi n_0 M}$ be fulfilled. For $\bar{\nu} \to 0$, the maximum amplitude achieved at the end of the first half period of fluctuations, corresponds to /63

$$H_{max} = 4u\sqrt{\pi n_0 M} - H_0.$$

It is not possible to determine $H(x)$ in a clear form. However, if the damping over one period is small, it is possible to use a simple approximate method of slowly changing amplitude, averaging over rapid oscillations. In the absence of friction, the movement of "a particle" in our potential hole is determined with the aid of one constant c which represents the level of total energy of the "particle" (see Figure 14). Then the determination of the reverse dependence of x on H is reduced to the square

$$\int \frac{dH}{\sqrt{(H-H_0)^2\left[1-\frac{(H+H_0)^2}{16\pi n_0 M u^2}\right]+C}} \pm \frac{x}{a}. \qquad (64)$$

The solution of the corresponding problem will be $H = \Phi(x, C)$. In the

method of slowly changing amplitude, let us write the solution (taking friction into account) in the form $H = \Phi(x, C_x)$, but C is now considered to slowly decrease with an increase in x (due to energy "dissipation"). The dependence of C on x is determined after averaging over the period by the following equation:

$$\frac{dc}{dx} = \frac{\frac{\bar{v}}{u}\int_{\Phi_1}^{\Phi_2}\sqrt{(\Phi-H_0)^2\left[1-\frac{(\Phi+H_0)^2}{16\pi n_0 M u^2}\right]+c}\,d\Phi}{\int_{\Phi}^{\Phi_2}\left(\sqrt{(\Phi-H_0)^2\left[1-\frac{(\Phi+H_0)^2}{16\pi n_0 M u^2}\right]+c}\right)^{-1}d\Phi}. \tag{65}$$

Here $\Phi_{1,2}$ represents two positive roots of the equation

$$(\Phi-H_0)^2\left[1-\frac{(\Phi+H_0)^2}{16\pi n_0 M u^2}\right]+C=0, \tag{66}$$

which are larger than H_0. Thus, the problem is reduced to solving equations (64 and (65). For $x \to \infty$, we have the boundary condition $H \to H_0$, $\frac{dH}{dx} \to 0$, i.e., $C \to 0$. For small C, both equations have simple asymptotic solutions. Thus, $\Phi(x, 0)$ has the form

$$\Phi(x, 0) \approx H_0\left[1+2(\mathcal{M}^2-1)\operatorname{sh}^2\frac{x}{a}\sqrt{\mathcal{M}^2-1}\right],$$

which coincides, naturally, with the profile of an isolated wave having small amplitude (see Section 2), where $\mathcal{M}=\left(\frac{4\pi M n_0 u^2}{H_0^2}\right)^{1/2}$ - i.e., the magnetic Mach number. For $C \to 0$ equation (65) is reduced to the following

$$\frac{dC}{dx} \approx -\frac{4}{15}\frac{\bar{v}}{u}\cdot\frac{H_0^2 8(\mathcal{M}-1)^3}{\ln\frac{\sqrt{-c}}{H_0\sqrt{\mathcal{M}^2-1}}}; \tag{67}$$

we thus find that

/64

$$C\ln\frac{\sqrt{-c}}{H_0\sqrt{\mathcal{M}^2-1}} \approx -\frac{4}{15}\frac{\bar{v}}{u}8H_0^2(\mathcal{M}-1)^3 x+\text{const}. \tag{68}$$

For large C, when the amplitude of the fluctuations considerably decreases as compared with the initial amplitude, the fluctuations become damped harmonics

$$H-H^* \sim e^{\frac{\bar{v}}{u}x}\sin\left(\sqrt{\mathcal{M}-1}\,\frac{x}{a}\right).$$

The profile of the change of H within the front of the wave can be finally represented in the following way (Figure 15). First, an

isolated wave appears in an undisturbed plasma, at the crest of which the magnetic field reaches a maximum value. Due to the presence of non-reversible dissipation (friction), the state of the plasma after the passage of such a wave will differ little from the original state. At a distance on the order of

$$\delta \approx \frac{a}{\sqrt{\mathcal{M}-1}} \ln \frac{u}{\bar{\nu}a} \sqrt{\mathcal{M}-1} \qquad (69)$$

a second wave moves after the first wave, etc. If we are not interested in the fine structure of the oscillations at the front of the shock wave and if we average over distances which exceed δ, then we can speak of δ as the effective thickness of the front of the shock wave connecting two states of the plasma: the undisturbed (before the arrival of the wave) and the disturbed (modulated by intense fluctuations), whose contribution upon the arrival must be included in the laws of conservation at a "discontinuity". In this sense, the role of damping is actually purely symbolic, since damping enters in the argument of the logarithm (Ref. 22) into the expression for δ (69) (the width of such a shock wave).

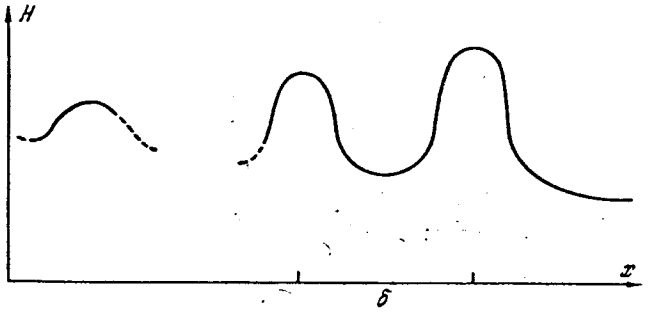


Figure 15.

The picture of damping of non-linear oscillations behind the front of the shock wave has the following character. In subsequent "isolated" waves, the amplitude decreases, and the distance between the two adjacent "elevations" in the magnetic field is reduced to /65

$\frac{a}{\sqrt{\mathcal{M}_b - 1}}$, when a set of elevations and depressions becomes a damped sinusoid. The total damping distance of the oscillations will be on the order of Δ

$$\Delta \sim \frac{u}{\bar{\nu}}. \qquad (70)$$

For Mach numbers which are not too close to unity, this formula cannot be used, since $\bar{\nu}$ will change within the front of the wave. But in order to estimate Δ, it is possible to use simply expression $\frac{u}{\langle \nu \rangle}$,

where $\langle\nu\rangle$ is the mean frequency of collisions between electrons and ions. Using $\lambda = \frac{\bar{v}_e}{\bar{\nu}}$ to express the mean free path ($\bar{v}_e$ is the mean relative velocity $\bar{\nu}$ of electrons with respect to ions), we obtain

$$\Delta \sim \sqrt{\frac{H^2}{8\pi nT} \cdot \frac{m}{M}}\lambda, \qquad (71)$$

It can thus be seen that even the damping distance of the oscillations due to collisions under real conditions, which correspond to the assumed approximation, can be significantly less than the free path (if $\frac{H^2}{8\pi nT}\frac{m}{M} \ll 1$). This is natural, since for Joule heating the temperature of the electrons increases, and their relaxation time - as a rule - is considerably less than that for ions due to greater velocity. "Equalization" of the ion and electron temperatures will take place after that, as the fluctuations within the front damp out, at distances on the order of $\lambda\left(\frac{M}{m}\right)^{1/2}$.

With the change to large Mach numbers, except for the limitations noted above, one still appears. It consists of the fact that, if the plasma in an undisturbed state is "cold", it is ultimately heated, so that the Larmor radii of the electrons become comparable with the characteristic oscillation length $\frac{c}{\omega_0}$. It is apparent that this occurs when the pressure of electrons becomes comparable with the magnetic pressure $nT \sim \frac{H^2}{8\pi}$. However, the picture for the initial stage of the front will have the above described character for Mach numbers which are not small. On the phase plane (H', H) the oscillating solution being investigated for the profile at the front of the shock wave has the form which is shown in Figure 16 (compare this with the corresponding integral curves in the absence of damping in Figure 5b). /66

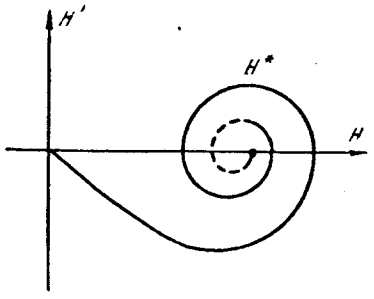


Figure 16.

It is interesting to establish a connection between the solution obtained for a rarefied plasma and the known expression for the thickness of a shock wave in magnetic hydrodynamics of a plasma for the analogous case of a weak wave propagated across a magnetic field

$$\Delta \sim \frac{\eta_m}{u(\mathcal{M}-1)}, \qquad (72)$$

where η_m is the so-called magnetic viscosity ($\eta_m = \frac{c^2}{4\pi\sigma}$, $\sigma = \frac{ne^2}{m\nu}$). The point $H = H^*$ is a special point of equation (62). Up until the present, in examining a rarefied plasma, we have tacitly assumed that damping [the role of the last term in equation (62)] is small, and the point H^* is automatically the focal point. In a dense plasma, the characteristic H^* becomes a "node" (Figure 17) under the condition

$$\frac{c}{\omega_0} < \frac{c^2 m\nu \cdot 1}{4\pi ne^2 u \cdot \sqrt{6}} \cdot \frac{H_0^{1/2}}{(H^* - H_0)^{1/2}}. \qquad (73)$$

In contrast to the profile investigated in the limiting case, when

$$\frac{c}{\omega_0} \ll \frac{c^2 m\bar{\nu}}{4\pi ne^2 u}$$

the known hydrodynamic profile occurs, which is determined by the magnetic viscosity. The thickness of the front is then given by expression (72).

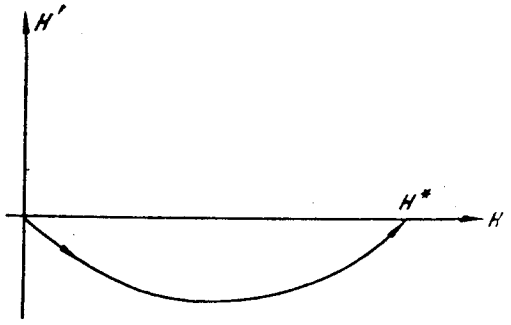


Figure 17.

2. To what effects can the small "non-collision" damping be attributed, due to the frequencies whose velocity is close to the propagation velocity of a shock wave (Ref. 14, Ref. 23)? The damping mechanism consists of energy being given to the ions which are reflected from the "humps" of the potential at the front of the shock wave (Figure 18). The magnetic field does not play a significant role in this reflection, if the Larmor radius of the ions is significantly greater than the characteristic dimension of the oscillations $\frac{c}{\omega_0} \cdot \frac{1}{\mathcal{M}-1}$.

The main portion of the reflection takes place at the first isolated wave (if, in general, the turning action of the magnetic field and the collisions is not taken into account, reflection would take place only at the first isolated wave). A quantitative determination of the effect of the reflected ions is rather cumbersome, and we did not carry this out (such a determination will be given below in the simpler case

/67

of a wave in the absence of a magnetic field).

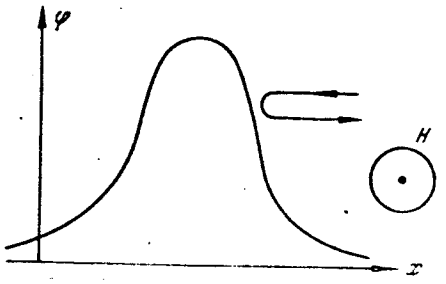


Figure 18.

We would like to note the curious effect which is produced by the acceleration of a group of ions in such a shock wave. The ions, whose velocity is quite close to the velocity of the shock wave, will have a very small Larmor radius. Therefore, after being reflected from the hump of the potential, they will again be immediately returned by the magnetic field, will again be reflected, etc. After repeated reflections (Figure 19), they acquire a very large velocity along the y-axis (in the plane of the front and across H). However, this velocity cannot become as large as desired, since as v_y increases the Lorentz force $\frac{e}{c} v_y H$ becomes significant in the region of the hump; ultimately, this force exceeds the "reflecting" force $-e\nabla\varphi$, and the ion crosses through the hump. With respect to the order of magnitude, the maximum energy of such ions equals $\frac{M}{m} \cdot \frac{Mu^2}{2}$, where $\frac{Mu^2}{2}$ is the mean energy of ordered motion per one ion in such fluctuations.

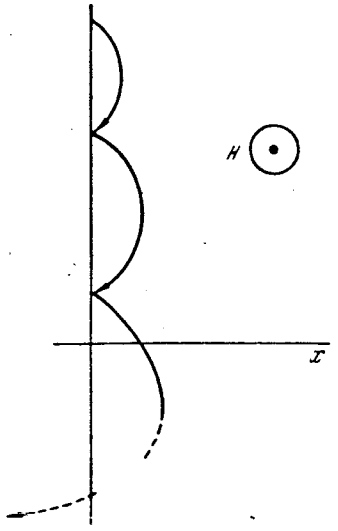


Figure 19.

Instabilities which lead to the energy transfer of ordered oscillations into energy of random fluctuations could serve as another mechanism for non-collision damping. Let us employ the results employed in the preceding section. The type of instability having the most transmittance for non-linear waves in a magnetic field is the "bunched" type, when the mean ordered velocity of the electrons with respect to the ions exceeds the mean thermal velocity $\left(v_0 > \sqrt{\frac{T}{m}}\right)$. This condition begins to be fulfilled for waves with a Mach number which exceeds

$$\mathcal{M}^* \approx 1 + \frac{3}{8}\left(\frac{8\pi nT}{H^2}\right)^{1/3}$$

[see formula (50)]. In physical terms, this instability signifies that electrons, moving with respect to the ions, are retarded not only due to normal collisions [the last term in equation (62)], but also due to unusual friction force of a collective nature - coherent "radiation" of plasma fluctuations due to the instabilities. For purposes of approximation, the following considerations can be employed: in the expression /68 for the electrical conductivity coefficient $\sigma_{eff} \sim \frac{ne^2}{m\nu}$; ν must now designate the inverse time for the loss of energy by electrons due to instability. Instead of $1/\nu$, it is reasonable to select in order of magnitude the effective time for the buildup of the instability, i.e.

$$\sigma_{eff} \sim \frac{ne^2}{m\Omega_0}. \qquad (74)^1$$

If the condition $\mathcal{M} > \mathcal{M}^*$ is fulfilled, the leading part of the front of the shock wave will be determined by such a damping effect. In terms of the effective potential well $V(H)$, the structure of the front will qualitatively have the form shown in Figure 20. At first, the sharp retardation of a particle is connected with the effect of instability. Then, as the amplitude of the pulsations decreases (and the temperature increases), the instability disappears, and further retardation is slowed down (Ref. 17). In reality, the damping of the oscillations in this region can be anomalous due to the decay instability.

3. Up to the present, we have been discussing the structure of a wave propagated in a "cold" plasma strictly across a magnetic field. Now we can readily extend the preceding investigation to the case of waves which are not perpendicular to H. The dispersion effects are very sensitive to the direction of wave propagation. If the wave is

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[1] This means that, if the condition of instability $v_0 > \sqrt{\frac{T}{m}}$ is fulfilled, anomalous electric resistance appears which leads to anomalous dissipation. This phenomenon was determined experimentally in waves having a large amplitude in a plasma placed in a magnetic field (Ref. 24).

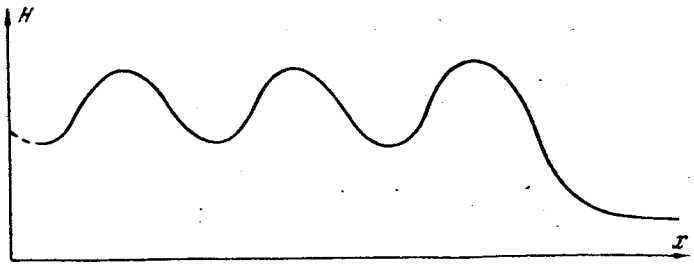

Figure 20.

not propagated absolutely transversely, the dispersion equation connecting ω with k takes the form of equation (20) in Section 2. The characteristic length of the dispersion is $\frac{C}{\Omega_0}\theta$ $\left(\text{for } \sqrt{\frac{m}{M}} \ll \theta \ll 1\right)$. The inertia of electrons for such waves is insignificant, but in return the gyrtropism of the plasma must be taken into consideration. The original system of equations for this case /69

$$M\frac{d\mathbf{V}}{dt} = e\mathbf{E} + \frac{e}{c}\mathbf{V}\cdot\mathbf{H};$$

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{V} = 0;$$

$$-e\mathbf{E} + \frac{e}{c}\mathbf{v}\times H = 0, \quad \frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v} = 0;$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},$$

$$\operatorname{rot}\mathbf{H} = \frac{4\pi en}{c}(\mathbf{V}-\mathbf{v})$$

can be reduced to the form

$$\left.\begin{aligned} \varrho\frac{d\mathbf{V}}{dt} &= -\nabla\frac{H^2}{8\pi} + \frac{(\mathbf{H}\nabla)\mathbf{H}}{4\pi};\\ \frac{\partial\varrho}{\partial t} + \operatorname{div}\varrho\mathbf{V} &= 0;\\ \frac{\partial\mathbf{H}}{\partial t} &= \operatorname{rot}\mathbf{V}\times\mathbf{H} - \frac{Mc}{e}\operatorname{rot}\frac{d\mathbf{V}}{dt}. \end{aligned}\right\} \qquad (75)$$

The component $\frac{Mc}{e}\operatorname{rot}\frac{d\mathbf{V}}{dt}$ gives the deviation from the linear law of dispersion for large k. The stationary solution of this system of equations (which, similarly to the preceding case, must include the Joule dissipation) describes the profile of the shock wave. For $\sqrt{\frac{m}{M}} \ll \theta \ll 1$, the equation describing the profile of the change in the magnetic field within the wave will take the form (Ref. 17, Ref. 25)

$$\frac{c^2}{\Omega_0^2}\theta^2\frac{d^2H_z}{dx^2}=H_z\left\{1+\frac{H_0^2}{8\pi\rho_0u^2}-\frac{H^2}{8\pi\rho_0u^2}\right\}-H_0+\alpha\frac{dH}{dx}. \qquad (76)$$

Not only the length of the dissipation changes here (instead of $\frac{c}{\omega_0}$, it becomes $\frac{c}{\Omega_0}\theta$), but also the nature of the dispersion ($\frac{\omega}{k}$ is increased with an increase in k). In comparison with equation (62), the sign of the "effective mass" is changed in equation (76). If damping $\alpha\frac{dH}{dx}$ is disregarded, equation (76) describes non-linear periodic steady waves. Isolated waves (Figure 21) represent a special solution here, but in this case this is isolation of a wave of "rarefaction". The profile of the shock waves will have the form shown in Figure 22. It is interesting to note that within the front of the shock wave the magnetic field reaches a minimum value which is less than in an undisturbed plasma.

The damping distance of such oscillations due to the normal friction force in order of magnitude equals /70

$$\Delta\sim\lambda\theta\left(\frac{H^2}{8\pi nT}\right)^{1/2}. \qquad (77)$$

The most outstanding difference from the preceding case consists of the fact that the leading edge of the oscillations is not sharp. For this reason, the impression can be formed that there is no shock wave without collisions, since λ (the mean free path) enters into expression (77) for Δ (the damping distance of the oscillations). However, the law of dispersion $\omega(k)$ for these fluctuations is exactly such, that the non-linear periodic waves are unstable with respect to "decay" (Ref. 26) (see Section 2). Due to the decay instability, non-linear

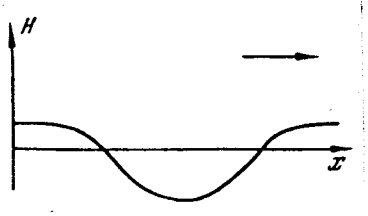

Figure 21.

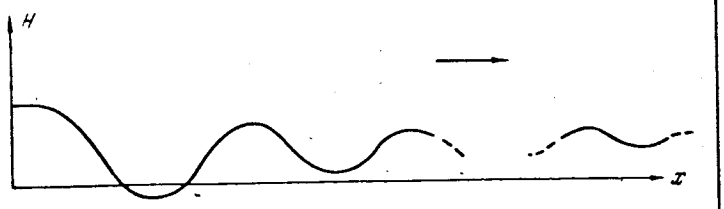

Figure 22.

ordered fluctuations damp out much sooner than would be the case according to formula (77), since their energy is changed into energy of an entire noise spectrum. The damping distance Δ which is thus obtained can be identified with the width of the shock wave front. In order to determine Δ it is necessary to find the level of the noises which arise due to decay instability, and to determine their reverse effect on the background. This is an extremely complex problem which has

still not been solved[in one of the works (Ref. 8)an attempt was made to obtain an approximate estimate of Δ]. Following a dimensional line of reasoning, it can be expected that Δ must be on the order of several oscillation lengths (except for wave lengths of oscillations within the front, there are no other characteristic dimensions in the problem).

Thus, the problem regarding the laminar structure of non-linear oscillations within the front of shock waves can be reduced to two different formulations (Figure 23): 1) when the dispersion curve $\omega(k)$ has the form of type *1* (waves perpendicular to H in a cold plasma). In this case the leading edge of the oscillations is sharp (everything commences from the isolated waves), and one can speak about a non-collision shock wave even in laminar theory. 2) When the short waves have a propagation velocity which is larger than the long ones (type *2*). In this case, the leading edge of the oscillations /71
is extended, since the short waves pass the front. Anomalous damping of such oscillations is necessary here for the existence of a non-collision shock wave. Decay instability (which is inherent to spectra of type *2*) can be the cause of the damping. As a result of the development of such instability, the plasma changes to a turbulent state. The theory for the width of the shock waves in the first case is quite simple, but, in the second case, the construction of a quantitative theory is an extremely complex problem. However, the main mechanisms are qualitatively quite clear.

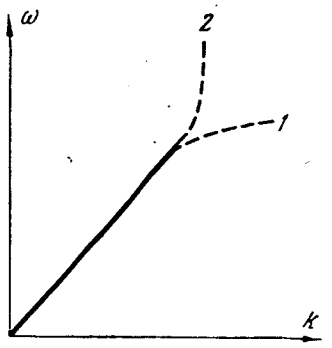

Figure 23.

It should be possible to give examples of problems which can be reduced to one of the two cases being analyzed. Thus, a shock wave across the magnetic field in a plasma of "large" pressure $\left(p \gtrsim \frac{H^2}{8\pi}\right)$ belongs to the second case, since the corresponding dispersion dependence (see Section 2) belongs to type *2*. Ion sound in a non-isothermic plasma ($T_e \gg T_i$) has a type *1* spectrum; therefore, the problem concerning a non-collision shock wave can be resolved in the laminar formulation.

4. In order to establish the profile of the front, in this case we shall use an analogy with the preceding paragraph. In the absence of a magnetic field, non-linear steady fluctuations exist for $T_e >> T_i$ if damping is disregarded, the equation describing the profile of the potential φ in such a wave [see equation (37)] has the form

$$\frac{d^2\varphi}{dx^2} = 4\pi n_0 e\left(\frac{u}{\sqrt{u^2 - \frac{2e\varphi}{M}}} - e^{\frac{e\varphi}{T}}\right) = -\frac{dV(\varphi)}{d\varphi}, \qquad (78)$$

where $V(\varphi)$ is the effective potential energy. We shall assume that normal dissipation due to collisions between the particles is absent, but we shall take into consideration the effect of ions being reflected from the leading edge of the wave, which plays the role of non-collision dissipation.

The structure of a non-collision shock wave which is thus formed can be described by using the following clear picture. In the absence of any dissipation, an isolated wave - which represents a symmetrical hump of the potential - can be propagated in the plasma. In actuality, there is a small group of ions in the plasma which are reflected from the moving potential hump, which leads to a disturbance of the symmetry. /72
Periodic fluctuations appear behind the hump, so that as a result an unusual shock wave is formed which connects two different states of the plasma: the undisturbed state (before the front) and a state modulated by intense ordered fluctuations (behind the front). The corresponding "impact adiabat" must take into account the additional components in the expressions for densities of the streams of energy and impulse behind the front, which are connected with the ordered fluctuations. However, it should be noted that the energy distribution between the thermal motion and the fluctuations depends on the concrete mechanism of "non-collision" dissipation. If the number of reflected particles is small, then the form of the profile of such a shock wave can be found. The form of the potential in the wave is shown in Figure 24. In the absence of dissipation $\varphi_1 = \varphi_2$ and $\lambda = \infty$ is the symmetrical isolated wave.

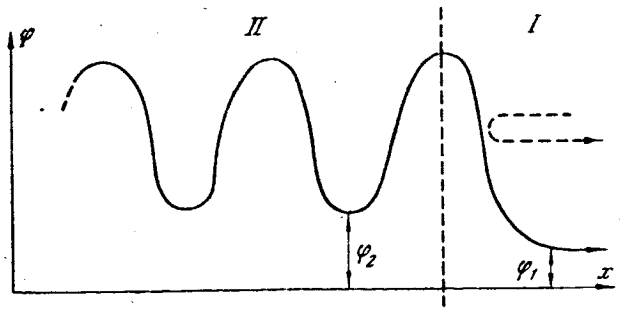

Figure 24.

In a consideration of the reflected ions, for the potential in region I (see Figure 24) there is an equation which differs from the equation (78) by the presence of additional components in the right side:

$$-4\cdot n_0 ef(\varphi_1)\frac{u}{\sqrt{u^2-\frac{2e\varphi}{M}}},\quad 2\cdot 4\pi n_0 ef(\varphi).$$

The first component corresponds to the subtraction of the reflected ions from the total number of ions n_0; the second component represents the contribution of the reflected ion. The quantity $n_0 f(\varphi)$ is the total density of the reflected ions at a point with the potential φ (the explicit form of f can be readily determined, if the undisturbed ion distribution in terms of velocity is known).

The potential jump φ_1 is connected with the ions departing to infinity, which are reflected from the potential barrier. In the case under consideration of a small number of reflected particles ($f \ll 1$) is proportional to f, and the quantity φ_2 - as we shall see - is proportional to the square root or the number of reflected particles, so that $\varphi_1 \ll \varphi_2$. The state of the plasma behind the front (in region II) is characterized by the quantities φ_{max}, φ_2, which determine the amplitude of the fluctuations and the long oscillation λ In this region equation (78) is valid. /73

Solving the equation for the potential in the regions I and II with the boundary conditions of continuity φ and $\frac{d\varphi}{dx}$, we can find the profile of the potential. If we again turn to the analogy with the motion of a particle in the potential wall $V(\varphi)$, then it can be stated that the effect of the reflected ions can be reduced to the fact that the total energy C remains negative. This leads to periodic motion (to periodic structure behind the front of the wave).

A decrease in the energy C is proportional to the number of reflected ions

$$-C\sim\int_0^{\varphi_{max}} f(\varphi)\,d\varphi.$$

Since the potential energy $V(\varphi)$ changes quadratically for small φ, the reversal point φ_2 is proportional to the square root of the energy C

$$\varphi_2\sim\sqrt{-C},$$

and the fluctuation period increases logarithmically with a decrease in energy

$$\lambda \sim \ln \frac{1}{C}.$$

Thus, the minimum value of the potential φ_2 behind the front of the shock wave equals

$$\varphi_2 = \frac{2\mathcal{M}}{\sqrt{\mathcal{M}^2 - 1}} \left(\frac{T}{e} \int_0^{\varphi_{max}} f(\varphi)\, d\varphi \right)^{1/2} \qquad \left(\mathcal{M}^2 = \frac{u^2}{\frac{T}{M}} \right). \tag{79}$$

The value φ_{max} differs little from the corresponding value in an isolated wave with the same Mach number.

The length of the oscillations at the front is (Ref. 27)

$$\lambda = \frac{A}{\sqrt{\mathcal{M}^2 - 1}} \left(\frac{T}{\pi n_0 e^2} \right)^{1/2} \ln \frac{\varphi_{max}}{\varphi_2}, \tag{80}$$

where $A \sim 1$.

5. We have still not touched upon the cases when it is impossible, generally speaking, to construct the laminar theory of the front of a shock wave. In the examples already examined, such a situation exists for rather large amplitudes, when there are no steady non-linear waves. Let us first examine the case when a shock wave goes across a strong magnetic field in a cold plasma. Within the front of the shock wave, ordered fluctuations exist for small Mach numbers here. As the amplitude of the magnetic field in the wave approaches a value which is two times larger than the initial magnitude of the magnetic field, the ordered oscillator structure is disturbed. In fact, as follows from Section 2, /74 for such amplitudes (for Mach numbers which are greater than 2) an isolated wave does not exist. What is more, it is impossible to construct steady, non-linear flow of a one-velocity type, i.e., such that there is only one value of ion velocity at each point in space. In physical terms, this means that as soon as the amplitude of the wave reaches the critical value ($H_{max} = 3H_0$) "inversion" occurs. In a certain section of space, the ions which at first move along behind reach the leading edges, and pass them (Figure 25). At this moment the velocity profile becomes three-valued.

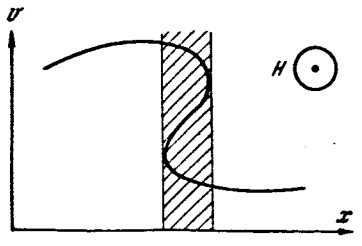

Figure 25.

We should note that the analogous phenomenon has been given a great deal of study in the theory of waves having finite amplitude on the surface of a heavy liquid in a canal having finite depth. Here there are non-linear steady motions of isolated - or periodic - wave types. At sufficiently large amplitudes, such waves break down due to "inversion". It is clear that a rigorous mathematical analysis of the picture which is formed after inversion is extremely difficult. Only a qualitative attempt may be made to establish the most significant characteristics of the phenomenon, by drawing an analogy with waves in a liquid.

The basic problem consists of determining whether the motion of the plasma would tend toward any stable regime after inversion, or whether the transitional region (which is cross-hatched in Figure 25) will be washed-out indefinitely, as would occur in a normal gas without collisions. In a theory of surface waves, for a certain period of time after inversion a steady flow is formed, which can be called "jump of water" or " bora ", which represents a certain transitional region of finite width, which is usually replaced by an idealized mathematical surface dividing two plane-parallel streams. The corresponding laws of conservation are satisfied with transition through this surface. In the well-known sense, the " bora " represents the analog of the shock wave. The fact that the thickness of the transitional layer is stationary can be physically explained by the fact that the sections of the profile - which push forward during inversion, ultimately circumscribing each other - drop back under the influence of gravity and "are mixed" with those which are at rest. The magnetic field, which turns the ions, plays the role of gravity in a plasma. Although their velocity distribution is far from a Maxwell distribution, it is possible to connect the plasma states on both sides of the transitional region with the laws of conservation of matter, impulse, and energy streams, using the quantity $\frac{M}{2}\overline{(v-\bar{v})^2}$ to designate the energy of thermal motion (the bar indicates averaging with respect to velocity distribution). The width of the transitional region can be estimated as the radius of ion curvature after inversion in the magnetic field (Ref. 22). Since the velocity amplitude is $v \gtrsim \frac{H}{\sqrt{4\pi\rho}}$ in a wave having a Mach number which is greater than 2, the width of the transitional layer (or the width of a non-collision shock wave) will be on the order of

/75

$$\delta \sim \frac{vMc}{eH} \sim \frac{c}{\Omega_0}, \quad \left(\Omega_0^2 = \frac{4\pi ne^2}{M}\right). \tag{81}$$

Multi-velocity flow - which arises after inversion - with velocities which are perpendicular to the magnetic field must be unstable, however. In fact, if - for purposes of simplicity - two-bunched distribution of ions with a velocity difference between the bunches which exceeds $\sqrt{\frac{T}{M}}$,

is investigated, then an instability arises with respect to the build-up of fluctuations which has a wave vector almost parallel to the velocity of the bunch. Instability of a similar type (counter currents) also occurs in the "bora ". This is simply instability of tangential discontinuity, which is formed when a falling stream comes in contact with the surface of a liquid at rest.

If the characteristic dimensions of the regions of multi-current motion considerably exceed the wave lengths of the instabilities which are formed, it is possible to use the results obtained in a study of stability in a homogeneous plasma. For example, in the case of two counter ion streams, moving across a magnetic field with the velocities v_0 and $-v_0$, the dispersion equation has the form (Ref. 28)

$$\frac{2}{\omega_{H_i}\omega_{H_e}} = \frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}. \qquad (82)$$

In order of magnitude, the maximum increment of instability equals $(\omega_{H_i}\omega_{H_e})^{\frac{1}{2}}$. For $v_0 \lesssim \frac{H}{\sqrt{4\pi nM}}$, the characteristic wave length of instability is on the order of $\frac{c}{\omega_0}$. Thus, multi-current motions across a magnetic field are unstable.

If the plasma were "hot" - i.e., if it were necessary to take the velocity scatter into consideration - then the dispersion equation in the form (82) would not be valid. For $v_{T_i} \sim \frac{H}{\sqrt{4\pi\rho}}$ the maximum increment of instability is equal in order of magnitude to ω_{H_i}, and the corresponding wave length - $\frac{c}{\Omega_0}$. Consequently, this quantity /76
characterizes the width of the front of a shock wave in a strong magnetic field[1].

6. We have investigated above the problem of non-collision turbulent shock waves in a plasma which are propagated across a strong magnetic field. A magnetic field, which is parallel to the plane of the

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[1] The work (Ref. 29) discusses the numerical calculation of a one-dimensional model for the motion of a plasma across the magnetic field at large amplitudes, when the flow becomes a multi-current one (and unstable). In particular, for $\mathcal{M}_0$=5.8 the effective "mixing length" - width of the front is 3.4 r_i.

A. Kantrowitz and H. Petschek (Ref. 7) have developed a semi-phenomenological theory of the turbulent structure of the front of a shock wave propagated across a magnetic field. The authors assumed that - due to some unknown instability in the plasma - an entire spectrum of fluctuations exists from the very beginning, and the interaction between different harmonics of this spectrum determines the processes of energy and impulse transfer.

wave front, retains the "hotter" particles, preventing the erosion of the transitional region between the disturbed (cold) plasma (before the front of the shock wave) and the hot plasma behind the wave. The possibility of the existence of non-collision shock waves in a plasma without a magnetic field has been discussed in several works. The so-called "bunched" instability of two inter-penetrating plasmas (Ref. 30) has been indicated here as the mechanism for restriciting the erosion of the transitional region. The thermal scatter within each of the "bunches" is not taken into consideration, however, in such an approach. But a more rigorous investigation, which would take the thermal motion into account, does not give instabilities up to Mach numbers from one to approximately $\left(\frac{M}{m}\right)^{1/2}$, if the electron temperatures are comparable with the ion temperature or smaller [M - ion mass; m - electron mass; see, for example, work (Ref. 31)].

As can be shown, in a non-isothermic plasma ($T_e >> T_i$), such a problem does not exist, since there it is possible to construct a laminar theory. However, for $\mathcal{M} > 1.6$ a different solution must be sought here due to inversion. This solution consists of utilizing the well-known anisotropic instability. When the more rapid ions from the region behind the front fall into the undisturbed plasma in front of the front, the ion distribution, with respect to the velocity in it, becomes non-isotropic. Such a state of the plasma is unstable. Random fluctuations of the electric and magnetic fields arise. The width of the shock wave front must then be designated by a quantity on the order of the mean free path of the ions, with respect to the scatter in such unbalanced fluctuations. This problem was solved in work (Ref. 27) with the same degree of "exactness" which was considered permissible in the theory of turbulence. We shall present some graphic, qualitative estimates here.

Let $H = 0$ in an undisturbed plasma. Let us start by understanding the physical meaning of anisotropic instability in this case. Thus, let /77
us take a plasma with different mean particle energies, in the x and y direction, for instance [$\varepsilon_{x,y} = M\overline{(v - \bar{v})^2}$, while $\varepsilon_y > \varepsilon_x$]. Let us investigate a disturbance which has the form of a magnetic field fluctuation which is as small as desired; let us direct it, for example, along the z-axis (Figure 26). Non-isotropic distributions can lead to its increase. In fact, let us examine particles moving along the y-axis close to the point x_0, where the magnetic field changes sign. The Laurentz force $F_x = \frac{e}{c} H v_y$ acts upon the particles. The particles having $v_y > 0$ will be pushed by this force to x_0, and the particles with $v_y < 0$ - to the side of x_0. Thus, particles with $v_y > 0$ will tend to concentrate close to x_0. This means that there is an electric

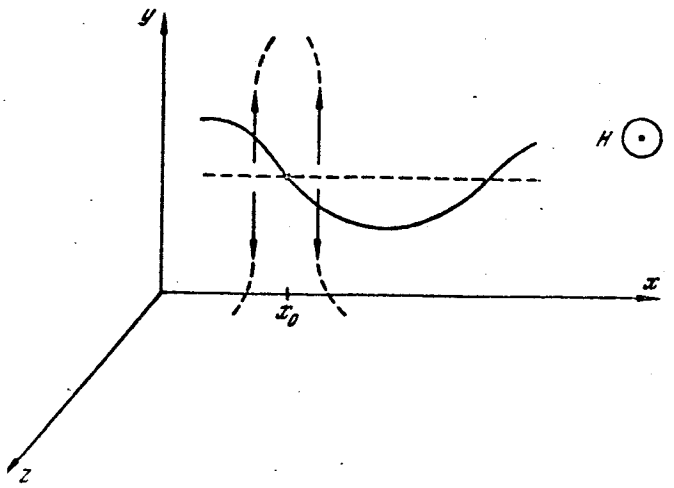

Figure 26.

current j_y. It is not difficult to analyze its direction, in view of the fact that it leads to an increase in the original fluctuation of the magnetic field, i.e., to instability. However, we did not take into consideration the stabilizing role of thermal motion along the x-axis, which counteracts such "condensation" of particles with similar signs v_y. If $\varepsilon_x = \varepsilon_y$ then there is no instability in general. If $\varepsilon_y > \varepsilon_x$, then for sufficiently long waves this counteraction is not in a condition to cancel out the instability. The order of magnitude of the wave length of such disturbances can be readily estimated. Let us compare two forces: the Laurentz force, which tends to result from equilibrium, and the counteraction, which - for purposes of approximation - can be taken in the form of a pressure gradient along x. The following is necessary for instability:

$$\frac{e}{c} v_y n_0 \delta H > \operatorname{grad} M\bar{v}_x^2 \delta n, \tag{83}$$

where δH and δn are the fluctuations of the magnetic field and density. On the other hand, the quantities δH and δn are connected with the Maxwell equation

$$\operatorname{rot} \delta H \sim \frac{4\pi}{c} e v_y \delta n. \tag{84}$$

Representing δH and δn in the form $\sim e^{ikx}$, with the aid of equation (84) we can rewrite condition (23) /78

$$\frac{\Omega_0^2}{c^2} v_y^2 > k v_x^2,$$

Thus, assuming that v_x^2 and v_y^2 are quantities of one order of magnitude, we obtain the characteristic k

$$k^2 < \frac{\Omega_0^2}{c^2}. \tag{85}$$

where $\Omega_0^2 = \frac{4\pi ne^2}{M}$. It is obvious that the entire effect is connected with ions, i.e., the ions carry the basic energy.

Thus, let us say that a disturbance is formed in a certain region of a rarefied plasma. In the absence of any restricting mechanism, the disturbance would be diffused with the passage of time in connection with the departure of the more rapid particles. In fact, when the particles arrive at new regions, velocity distribution anisotropy is formed there, along with the instability which is connected with this. The disordered magnetic field which arises must apparently scatter the particles, imitating "collisions". Thus, the possibility arises of extending the non-diffused disturbance which has the nature of a shock wave, just as in normal gas dynamics.

Let us estimate the pulsation amplitude δH in a non-linear regime of instability which has developed. It can be expected that the fluctuation of the magnetic field will increase up until almost all of the energy excess of the ions $n\Delta\varepsilon$ (due to anisotropy) is changed into the energy of the magnetic field $\frac{(\delta H^2)}{8\pi}$ (for purposes of simplicity, we shall assume that $\Delta\varepsilon \sim \varepsilon \sim T$). However, the electrons - which could not be taken into consideration up until now - have a quenching action, "freezing" $(\delta H)^2$ at a much lower level. Actually, as only the mean Larmor radius of the electrons becomes on the order of $\lambda \sim \frac{1}{k}$ - the wave length of disturbances, which characterizes the spatial heterogeneity of the magnetic field - the electrons "are frozen" in the magnetic field. A further increase in the magnetic field must be accompanied by an enormous increase in electron energy in view of the retention of the adiabatic invariant $\mu = \frac{mv_\perp^2}{2H}$. Thus, it is reasonable to estimate δH from the condition $r_{H_e} \sim \frac{1}{k}$ which gives

$$\frac{(\delta H^2)}{8\pi} \sim \frac{m}{M} nT. \tag{86}$$

The scatter of ions in such a magnetic field will have a diffusion nature. The diffusion coefficient (in velocity space) can now be readily estimated as:

$$D \sim \frac{e^2}{M^2c^2} \cdot \frac{(\delta H)^2}{k} v_i. \tag{87}$$

We can thus find the ion "scattering" time $\tau \sim \frac{v_i^2}{D}$ and the corresponding mean free path $l \sim \tau \cdot v_i \sim \frac{M}{m} \cdot \frac{c}{\Omega_0}$. This will determine /79

in order of magnitude the width of the shock wave front

$$\Delta \sim \frac{M}{m} \cdot \frac{c}{\Omega_0} . \tag{88}$$

A "rigorous" theory leads to the following result:

$$\Delta \sim \frac{c}{\Omega_0} \cdot \frac{M}{m} \cdot \frac{1}{(\mathcal{M}-1)^2} \text{ for } M-1 \ll 1, \tag{89}$$

in which the dependence on the wave amplitude is also taken into consideration.

An analogous investigation can be made for a plasma in which there is a weak magnetic field $\left(\frac{H^2}{8\pi} \ll nT\right)$ from the very beginning. Beginning with $H \sim \delta H$, the shock wave front is contracted as H increases (Ref. 27, Ref. 32).

7. It can be hoped that the ideas regarding shock waves can be applied far beyond the limits of ordinary gas dynamics, based on an understanding of the mean free path with respect to pair collisions, since collective processes - plasma fluctuations - begin to play a primary role in a rarefied plasma. A single theory for the width of the front in a plasma, from which different partial cases could be automatically obtained, does not exist at the present time. The diversity of the phenomena connected with the collective processes is too great. Only very limited cases and procedures were summarized above, which would make it possible to understand any new element which appears here as compared with ordinary gas dynamics: dispersion effects, microscopic instabilities, non-collision damping, etc. Accordingly, in the different limiting cases there are very different lengths which characterize the wave front width (Debye radius, Larmor radius, $\frac{c}{\Omega_0} \frac{M}{m}$ etc.).

Unfortunately, at the present there have been almost no systematic laboratory experiments on shock waves in a rarefied plasma. Nevertheless, the separate effects which are utilized in constructing a theory of shock waves have recently been experimentally confirmed.

The phenomenon of the so-called sudden beginning of geomagnetic storms provides indirect proof of the theory. Even in 1955, T. Goyld drew the conclusion that the rapid increase of the earth's magnetic field (several minutes) in the first phase of a magnetic storm could be explained only by the fact that the solar flares generated shock waves in the interplanetary gas. Assuming that the ion density in the interplanetary plasma is $n \sim 10^2$ cm^{-3}, from formula (88) we can obtain the width of the shock wave front as on the order of 10^9-10^{10}cm which gives a characteristic time of 1 minute at a velocity of 10^8cm/sec. /80

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COULOMB COLLISIONS IN A FULLY IONIZED PLASMA

D. V. Sivukhin

1. DIFFICULTIES OF A THEORY OF COULOMB COLLISIONS

The construction of a systematic theory for collisions in a plasma encounters a great many difficulties, which are related to the slow decrease in the Coulomb forces with an increase in the distance between the interacting particles. At any moment in time, each particle of the plasma is acted upon by an enormous number of surrounding particles, and all of this action must be taken into account. Instead of the simple problem of two bodies, we arrive at the more difficult problem of many bodies. Such a problem is barely solvable in a rigorous formulation. In order to make a solution possible, it is necessary to introduce several simplifications. The simplest of these is the approximation of pair collisions, in which the interaction of plasma particles is reduced to the independent and instantaneous interaction of particle pairs. In this approximation, the continuous process - whereby each particle reacts with another particle in the plasma, which lasts indefinitely - is replaced by an instantaneous collision between these particles which leads to the same final change in their energy and impulses as that obtained during the actual process for an indefinite interaction time. Due to this fact, divergent integrals are obtained in calculating a different type of quantities. Their divergence can be eliminated by artificially cutting off the radius of the Coulomb forces action. This leads to results which coincide in all essential points with the results obtained by other, more systematic - although not devoid of the principal defects - methods for determining the interaction of many particles. In the present article, we shall confine ourselves to an approximation of pair collisions, and we shall therefore begin the discussion with an investigation of the problem regarding the collision of two particles. It will be shown in Section 19 that the assumption of finite interaction time in a scheme of pair interactions leads to the same results as are produced by an idealized scheme of pair collisions, where the interactions are assumed to be instantaneous. This Section can therefore be regarded as substantiation (although not strict) of the pair collision approach. /81 /82

Systematic methods for solving the problem concerned with unstable states are based on a *kinetic equation*. However, there are several problems which can be solved without utilizing a kinetic equation. This includes all problems dealing with stable or quasi-stable states, for which the distribution functions can be borrowed from statistics. These problems are of great interest. Their solution will entail the introduction of concepts and the production of results which are necessary for deriving a kinetic equation in the pair collision approach.

Therefore, we shall begin with an examination of this type of problem. Then, beginning with Section 14, we shall turn to the derivation of a kinetic equation and to several of its applications. Elementary methods are used in order to provide a better clarification of the physical bases of the theory. For this reason, we shall not investigate the following important studies which are relevant here: studies of N. N. Bogolyubov, V. P. Silin, O. V. Konstantinov and V. I. Perel', Rostoker, Balesku, Lenard, Yu. L. Klimontovich, S. V. Temko, and others.

2. COLLISION OF TWO PARTICLES

1. The *collision* of two particles is understood to mean the interaction of these particles when it is possible to disregard the time of this interaction and to assume that it is negligible. If the *interaction time* cannot be disregarded, then we shall use the more general term of "*interaction*" instead of the term "collision". In the problems dealing with collisions, we are interested in the relationship between the parameters determining the states of the colliding particles before and after collision. We can here establish such a relationship between these parameters, which is derived from the laws of impulse and energy conservation. It is assumed that the collisions are *elastic* , so that the internal state of the colliding particles does not change as a result of the collision.

We shall use m and m^* to designate the mass of the colliding particles, $\mathbf{v}$ and $\mathbf{v}^*$ - to designate their velocities before collision, $\mathbf{v} + \delta\mathbf{v}$ and $\mathbf{v}^* + \delta\mathbf{v}^*$ - to designate velocities after collision. According to the law of impulse conservation

$$m\delta\mathbf{v} + m^*\delta\mathbf{v}^* = 0. \tag{2.1}$$

If $\mathbf{u} = \mathbf{v} - \mathbf{v}^*$ is the velocity of the first particle with respect to the second, then

$$\delta\mathbf{v} - \delta\mathbf{v}^* = \delta\mathbf{u}.$$

Solving this equation together with equation (2.1), we find

$$\delta\mathbf{v} = \frac{m^*}{m+m^*}\delta\mathbf{u} = \frac{\mu}{m}\delta\mathbf{u},$$
$$\delta\mathbf{v}^* = -\frac{m}{m+m^*}\delta\mathbf{u} = -\frac{\mu}{m^*}\delta\mathbf{u}, \tag{2.2}$$

where μ is the reduced mass:

$$\mu = \frac{mm^*}{m+m^*}. \tag{2.3}$$

For an increase in $\delta\mathbf{p} = m\delta\mathbf{v}$ and $\delta\mathbf{p}^* = m^*\delta\mathbf{v}^*$ of the colliding

particle impulses, we obtain

$$\delta \mathbf{p} = -\delta \mathbf{p}^* = \mu \delta \mathbf{u}, \tag{2.4}$$

and for an increase in $\delta \mathcal{E}$ of the kinetic energy of the first particle

$$\delta \mathcal{E} = \frac{m}{2}(\mathbf{v} + \delta \mathbf{v})^2 - \frac{m}{2}\mathbf{v}^2 = m\mathbf{V}\delta \mathbf{v} + \mu \mathbf{u}\delta \mathbf{v} + \frac{m}{2}(\delta \mathbf{v})^2,$$

where $\mathbf{V}$ is the center of inertia velocity of the system of two particles which is being considered:

$$\mathbf{V} = \frac{m\mathbf{v} + m^*\mathbf{v}^*}{m + m^*}, \tag{2.5}$$

and which does not change as a result of the collision.

It follows from the law of energy conservation - described within a frame of reference in which one of the colliding particles is at rest - that the vector length of the relative velocity $\mathbf{u}$ does not change during a collision either. This vector only changes with respect to direction. Therefore,

$$\delta (\mathbf{u})^2 = 2\mathbf{u}\delta \mathbf{u} + (\delta \mathbf{u})^2. \tag{2.6}$$

If this is taken into consideration along with relationship (2.2), we can readily find:

$$\begin{aligned} \delta \mathcal{E} &= m(\mathbf{V}\delta \mathbf{v}), \\ \delta \mathcal{E}^* &= m^*(\mathbf{V}\delta \mathbf{v}^*), \end{aligned} \tag{2.7}$$

where $\delta \mathcal{E}^*$ is the increase in the kinetic energy of the second particle. In view of relationship (2.1), $\delta \mathcal{E} + \delta \mathcal{E}^* = 0$, as must be the case according to the law of energy conservation.

Since

$$\begin{aligned} \mathbf{v} &= \mathbf{V} + \frac{m^*}{m + m^*}\mathbf{u}, \\ \mathbf{v}^* &= \mathbf{V} - \frac{m}{m + m^*}\mathbf{u}, \end{aligned} \tag{2.8}$$

then formula (2.7) can be reduced to the form

$$\delta \mathcal{E} = -\delta \mathcal{E}^* = \mu(\mathbf{V}\delta \mathbf{u}). \tag{2.9}$$

2. Let us express $\delta \mathbf{u}$ by the rotation angle ϑ of the vector $\mathbf{u}$, as a result of particle collisions. $\mathbf{u}'$ can be used to designate the vector of relative velocity after collisions. Its length equals the

length of the vector u. Therefore, as can be seen from Figure 1,

$$\delta u_{\parallel} = u(\cos\vartheta - 1) = -2u\sin^2\frac{\vartheta}{2},$$

$$\delta u_{\perp} = u\sin\vartheta, \tag{2.10}$$

where $\delta u_{\parallel}$ and $\delta u_{\perp}$ represent the values of the vector components δu along the vector u and perpendicular to it. In vector form

$$\delta \mathbf{u} = [\mathbf{ku}]\sin\vartheta - 2\mathbf{u}\sin^2\frac{\vartheta}{2}, \tag{2.11}$$

where k is the only vector which is perpendicular to the plane (u, u') and which forms a right-handed system with the rotation direction of the vector u. Thus,

$$\delta p_{\parallel} = -2\mu u\sin^2\frac{\vartheta}{2}, \tag{2.12}$$

$$\delta p_{\perp} = \mu u\sin\vartheta, \tag{2.13}$$

$$\delta \mathbf{p} = -\delta \mathbf{p}^* = \mu[\mathbf{ku}]\sin\vartheta - 2\mu\mathbf{u}\sin^2\frac{\vartheta}{2}, \tag{2.14}$$

$$\delta\mathcal{E} = -\delta\mathcal{E}^* = \mu\sin\vartheta([\mathbf{uV}]\mathbf{k}) - 2\mu\sin^2\frac{\vartheta}{2}(\mathbf{Vu}). \tag{2.15}$$

Finally, projecting the relationship (2.4) in the direction of the vector u of the relative velocity before collision, we obtain

$$\delta p_{\parallel} = \mu\frac{\mathbf{u}\delta\mathbf{u}}{u},$$

where $\delta p_{\parallel}$ is the projection of the vector δp in the direction of u.

With the aid of formulas (2.6) and (2.4), we can reduce this expression to the form

$$\delta p_{\parallel} = -\frac{\mu}{2u}(\delta u)^2 = -\frac{(\delta p)^2}{2u\mu}. \tag{2.16}$$

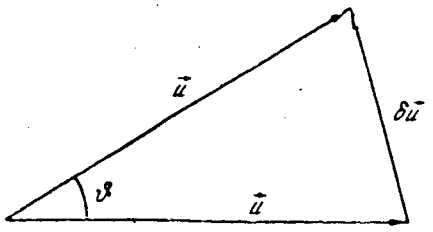

Figure 1.

3. All of the obtained relationships were derived from only one

of the *laws of energy and impulse conservation*, and therefore they are valid for *any law of power interaction* between colliding particles. They are valid both in *classical*, and in *quantum* mechanics, since they include no statements with regard to the spatial localization of colliding particles, and the particle states before and after collision are investigated - when the particles are infinitely removed from each other and, consequently, can have definite energy and impulse values.

3. MEAN VELOCITIES OF ENERGY AND IMPULSE CHANGE FOR A TEST PARTICLE IN A PLASMA. ELECTROSTATIC ANALOGY.

1. Let us now turn to an investigation of several problems which can be solved by utilizing a kinetic equation. It is assumed that external force fields are absent. /85

Let us assume that any particle with a mass of m a charge e, and a velocity $\mathbf{v}$ moves in a plasma. In terms of established terminology, such a particle is called a *test particle*. All other particles are regarded as the medium in which the test particle moves. They are called *field particles*. It is necessary to find the mean velocities of the change in energy $\mathcal{E}$ and impulse $\mathbf{p}$ of a test particle due to its interaction with the field particles. When speaking of the mean values of the physical quantities, we have in mind the *averaging with respect to a group of test particles*, i.e. a set of identical test particles which interact with each other, which move in a plasma for one and the same initial impulse value (and, consequently, energy). In the pair collision approach, changes in energy and impulse of a test particle are composed of changes in these quantities, which they undergo as a result of *independent* collisions of a test particle with each of the field particles.

2. Let us first examine the case in which all of the field particles are the same and move at one and the same velocity $\mathbf{v}^*$. Let n^* be the concentration of field particles, i.e., their number per unit of volume; m^* -the mass; e^*- the charge of the field particle. Within the frame of reference in which the field particles are at rest, a test particle moves at a velocity of $\mathbf{u} = \mathbf{v} - \mathbf{v}^*$. Upon each collision with the field particles, the energy and impulse of the test particle receives increases in $\delta\mathcal{E}$ and $\delta\mathbf{p}$, which are determined by expressions (2.15) and (2.14). As a result of the collision, the vector length of relative velocity of $\mathbf{u}$ does not change, and the vector itself $\mathbf{u}$ is turned at a certain angle ϑ . The probability that during time dt the vector $\mathbf{u}$ is deflected by at an angle ranging between ϑ and $\vartheta + d\vartheta$ - i.e., it falls within the solid angle $d\Omega = 2\pi \sin \vartheta d\vartheta$,equals $n^* u\sigma(\vartheta, u)d\Omega dt$ where $\sigma(\vartheta, u)$ is the

differential effective scattering cross-section of the test particle by a field particle - within a frame of reference where the field particles are at rest. Therefore, the mean energy and impulse increase of a test particle during time dt are determined by the expressions

$$n^* u\, dt \int \delta\mathcal{E}\sigma(\vartheta, u)\, d\Omega \quad \text{и} \quad n^* u\, dt \int \delta p \sigma(\vartheta, u)\, d\Omega,$$

and the mean velocities of their change are determined by the equalities

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = n^* u \int \delta\mathcal{E}\sigma(\vartheta, u)\, d\Omega, \tag{3.1}$$

$$\left\langle \frac{dp}{dt} \right\rangle = n^* u \int \delta p \sigma(\vartheta, u)\, d\Omega. \tag{3.2}$$

With the formulation of expressions (2.15) and (2.14), the components μ [ku] sin ϑ and μ sin ϑ ([uV]k) can not be taken into consideration, since the vector k can assume any direction which is perpendicular to the vector u with the same probability. Therefore /86

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = -4\pi\mu n^* u\, (\mathbf{Vu}) \int_0^\pi \sin^2 \frac{\vartheta}{2} \sigma(\vartheta, u) \sin\vartheta\, d\vartheta, \tag{3.3}$$

$$\left\langle \frac{dp}{dt} \right\rangle = -4\pi\mu n^* u \mathbf{u} \int_0^\pi \sin^2 \frac{\vartheta}{2} \sigma(\vartheta, u) \sin\vartheta\, d\vartheta. \tag{3.4}$$

3. For Coulomb collisions, the differential cross-section $\sigma(\vartheta, u)$ is given by the well-known *Rutherford law*

$$\sigma(\vartheta, u) = \left(\frac{ee^*}{2\mu u^2}\right)^2 \frac{1}{\sin^4 \frac{\vartheta}{2}}. \tag{3.5}$$

If this expression is substituted in formulas (3.3) and (3.4), then the divergent integral $\int_0^\pi \operatorname{ctg} \frac{\vartheta}{2} d$ is obtained. The divergence is due to inversion at infinity of the integrand at the lower limit $\vartheta = 0$. This is explained by the fact that the *real interactions* in all calculations are replaced by *instantaneous collisions*. In formulas (3.1) and (3.2), we have used $\delta\mathcal{E}$ and δp to designate the energy and impulse increase in a test particle throughout the *entire* period of time that it interacts with field particles, while according to the problem they would be used to designate the increases throughout the time dt entering into these formulas. For this reason, exaggerated values for $\delta\mathcal{E}$ and δp were used in the calculations. Such an exaggeration is particularly significant, when we are discussing the interaction of a test particle with *far-removed* field particles. It leads to divergence in the integral (3.3) and (3.4). If the correct values are

used for $\delta\mathcal{E}$ and δp[1], then finite expressions are obtained for the left-hand side in formulas (3.1) and (3.2). However, this does not dispose of the difficulties, since these finite expressions are physically unsuitable because - as the calculation shows - they distinctly contain the time integral dt . This question will be discussed in detail in Section 19.

In the theory of pair collisions, this difficulty is overcome by *artificially cutting off* the radius of action of the Coulomb forces. Each charged particle in the plasma attracts the particles having the opposite charge and repulses the particles having the same charge. Due to this fact, there is a tendency to form around it an excess number of particles having an opposite charge, which weakens (screens) its Coulomb field. Such a tendency can be calculated numerically, if it is assumed that the action of the Coulomb field for the particle extends /87
only to a distance which does not exceed a certain quantity D, and beyond that it is practically stopped. The *Debye radius* (see Section 4) is usually used as the quantity D . If the Debye radius is small as compared with the distance udt, which is traversed by the test particles during the period of time dt in the medium of field particles, then the interaction of this particle with each of the particles of the *Debye sphere* can be regarded as a collision. Then no divergence is obtained in the formulas (3.3) and (3.4), since the angle ϑ can not be less than a certain minimum limit $\vartheta_{\min}$. If this quantity is used as the lower limit of integration, then formulas (3.3) and (3.4) assume the form:

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = -\frac{4\pi n^*}{\mu u^3}(ee^*)^2 L(\mathbf{Vu}), \tag{3.6}$$

$$\left\langle \frac{d\mathbf{p}}{dt} \right\rangle = -\frac{4\pi n^*}{\mu u^3}(ee^*)^2 L\mathbf{u}, \tag{3.7}$$

where the following integral is designated by L:

$$L = \int_{\vartheta_{\min}}^{\pi} \operatorname{ctg}\frac{\vartheta}{2}\frac{d\vartheta}{2} = -\ln\sin\frac{\vartheta_{\min}}{2}, \tag{3.8}$$

which is called the *Coulomb logarithm*.

4. Let us now assume that there are certain types of field particles which differ from each other in terms of charge and mass. Let us introduce the *distribution function* $f^*(\mathbf{v}^*)$ for particles of any definite type. By definition, $f^*(\mathbf{v}^*)d\mathbf{v}^*$ gives the mean number of particles of the type under consideration per unit of volume,

[1] It is obvious that it is impossible to use the concept of differential scattering cross-section in the calculation, at least in the form in which it is usually introduced.

whose vector velocity extremes lie within the volume element $d\mathbf{v}^* \equiv dv_x^* dv_y^* dv_z^*$ of *velocity space* with the center at the point $\mathbf{v}^*$. Then from formulas (3.6) and (3.7), we can readily find

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = -4\pi e^2 \sum^* \int \frac{e^{*2}}{\mu u^3} L\,(\mathbf{Vu})\, f^*\,(\mathbf{v}^*)\, d\mathbf{v}^*; \tag{3.9}$$

$$\left\langle \frac{d\mathbf{p}}{dt} \right\rangle = -4\pi e^2 \sum^* \int \frac{e^{*2}}{\mu u^3} L\mathbf{u} f^*\,(\mathbf{v}^*)\, d\mathbf{v}^*. \tag{3.10}$$

An asterisk over the sum sign means that the summation was carried out for all types of field particles. We shall use this notation method from this point on in order to avoid utilizing indexes which block up the formulas.

The Coulomb logarithm L in the formulas (3.9) and (3.10) remains under the integral sign, since it can depend on the relative velocity of $\mathbf{u}$. Besides, this dependence is very weak. If it is disregarded and L is used to designate a certain mean (with respect to u) value of the Coulomb logarithm - which we shall do in the majority of the cases - then formulas (3.9) and (3.10) can be transformed as follows: /88

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = -\sum^* L \left\{ \frac{\mathbf{v}\mathbf{E}_v}{\mu} - \frac{\varphi_v}{m} \right\}; \tag{3.11}$$

$$\left\langle \frac{d\mathbf{p}}{dt} \right\rangle = -\sum^* \frac{L}{\mu} \mathbf{E}_v. \tag{3.12}$$

Here the following notations are introduced:

$$\mathbf{E}_v = \int \frac{\mathbf{u}}{u^3} \varrho_v\,(\mathbf{v}^*)\, d\mathbf{v}^*; \tag{3.13}$$

$$\varphi_v = \int \frac{1}{u} \varrho_v\,(\mathbf{v}^*)\, d\mathbf{v}^*; \tag{3.14}$$

$$\varrho_v = 4\pi\,(ee^*)^2 f^*\,(\mathbf{v}^*). \tag{3.15}$$

The quantities μ and L depend on the mass of both colliding particles, and therefore in formulas (3.11) and (3.12) they remain under the sum sign Σ^* (however, for L this dependence is very weak, and in the majority of cases it can be disregarded).

As regards the integrals (3.13) and (3.14), in terms of structure they are similar to the expressions for *intensity* and *potential* of an electric field in electrostatics. *Velocity space* plays the role of normal space. The quantity $\rho_v = 4\pi(ee^*)^2 f^*\,(\mathbf{v}^*)$ plays the role of *charge density* at the point $\mathbf{v}^*$ of this space. Vector $\mathbf{E}_v$ is formally similar to *intensity of the electric field* at the point $\mathbf{v}$, and the scalar φ_v is analogous to the *potential* of this field at the same

point. In order to emphasize this formal analogy, we designated these quantities by E_v and φ_v, providing them with the sign v - which indicates that we are not dealing with real electric fields, but with quantities which are their *analogs in velocity space*. This formal analogy was noted and systematically used in order to simplify the calculations of B. A. Trubnikov (Ref. 1), and also - independently of him it was used by Rosenbluth, MacDonald, and Judd (Ref. 2).

4. DEBYE SCREENING AND DEBYE RADIUS

1. The concept of *Debye screening* and *Debye radius* is very clear in the case when there is a fixed source, which creates an electrostatic field around itself, in a plasma. A balanced *Maxwell-Boltzmann distribution* of positive and negative ions in the plasma is established in this field. For the purposes of our article, it is sufficient to assume that the positive and negative ions in the plasma are *singly-charged* - the charge of each of them is numerically equal to the electron charge. We can designate the source charge by q. In the state of statistical equilibrium, an excess of ions having an opposite charge sign is formed around the charge q The potential of the average electrostatic field, created by the charge q and by the plasma ions surrounding it, can be designated by φ. In all of the space outside the charge q, the potential φ satisfies the *Poisson equation* /89

$$\Delta\varphi = -4\pi e(n^+ - n^-), \tag{4.1}$$

where e is the absolute value of the electron charge, and n^+ and n^- represent the concentration of positive and negative ions. For the Maxwell-Boltzmann distribution

$$n^+ = ne^{-\frac{e\varphi}{T}}, \quad n^- = ne^{\frac{e\varphi}{T}},$$

where T is the plasma temperature in energy units; n is the concentration of positive ions, or the concentration of negative ions which is equal to it to infinity. Thus,

$$\Delta\varphi = -4\pi ne\left(e^{-\frac{e\varphi}{T}} - e^{\frac{e\varphi}{T}}\right). \tag{4.2}$$

Let us examine the solution of this equation for two cases: when the charge q is a point charge, and when it is distributed equally along an infinite plane.

2. For a point charge, it is not possible to determine an accurate solution of equation (4.2) in a simple form. We shall confine ourselves to an approximate solution which is suitable for large distances from q, in which

$$|e\varphi| \ll T. \tag{4.3}$$

At such distances, the left-hand side of equation (4.2) can be expanded in power series, and this expansion can be terminated at the linear terms. In this approximation

$$\Delta\varphi + \frac{\varphi}{D^2} = 0, \tag{4.4}$$

where the expression is introduced

$$D = \sqrt{\frac{T}{8\pi ne^2}}. \tag{4.5}$$

In view of the spherical symmetry, the potential φ can depend only on the distance r from the point charge q, and equation (4.4) can be rewritten in the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) + \frac{\varphi}{D^2} = 0.$$

Its general solution is:

$$\varphi = \frac{1}{r}\left(Ce^{-\frac{r}{D}} + C'e^{\frac{r}{D}}\right). \tag{4.6}$$

The integration constant C' must equal zero, since the potential φ must become zero at infinity. The second integration constant C can be determined from the requirement that at small distances of r the solution of equation (4.2) changes into the *Coulomb potential* $\varphi = \frac{q}{r}$. Consequently, in order to determine C it is necessary to know, generally speaking, the form of the solution for equation (4.2) at small distances. However, if equation (4.3) is fulfilled even for $r \ll D$, then for such r the solution (4.6) is still applicable. But it then changes into $\varphi = \frac{C}{r}$, and we obtain the Coulomb potential, setting $C = q$. Thus, if the following condition is fulfilled, in addition to the condition (4.3):

/90

$$\left|\frac{eq}{D}\right| \ll T, \tag{4.7}$$

then $C = q$, and we obtain

$$\varphi = \frac{q}{r}e^{-\frac{r}{D}}. \tag{4.8}$$

This potential is called the *Debye potential*, and the distance D is the *Debye radius*. As can be seen from formula (4.8), in order of magnitude the Debye radius determines the distance from the charge q, at which the Coulomb field of this charge is screened oppositely by the charged ions of the plasma. It can also be stated that the influence of the Coulomb field of the charge q extends to a distance on the order of the Debye radius D, but at larger distances it is barely apparent.

3. It can be assumed that the requirement (4.7) is always fulfilled, under the condition that the macroscopic approach to solving the problem - which we have used - is applicable. Actually, in order that such a macroscopic approach be applicable, it is necessary that a very large number of plasma particles are contained within the *Debye sphere* (i.e., a sphere with the radius D). Otherwise, the concept of Debye screening and Debye radius loses its meaning - along with the averaging of the charge density and the electric field around the charge q, which was used in writing the Poisson equation (4.1). In mathematical terms, this requirement can be written as follows:

$$D^3 n \gg 1, \tag{4.9}$$

or, by taking expression (4.5) into account:

$$T \gg e^2 n^{1/3}. \tag{4.10}$$

This condition means that *the mean kinetic energy of thermal motion for a plasma particle must be large as compared with the potential energy of Coulomb interaction between it and a neighboring plasma particle, which is a distance of* $n^{-1/3}$ *away, which equals the mean distance between plasma particles having the same charge.* But, under condition (4.9), condition (4.7) is automatically fulfilled, if only the charge q (as it always is) coincides with e in order of magnitude. Actually, it follows from condition (4.9) that the Debye radius D is large as compared with the mean distance $n^{-1/3}$ between plasma particles having the same charge. But then the potential energy $\frac{eq}{D}$, which enters into condition (4.7), will be less than $e^2 n^{-1/3}$ in terms of absolute magnitude, and therefore inequality (4.7) follows from inequality (4.9) or (4.10). Thus, the validity of formula (4.8) is shown under the condition (4.10), and not only at large distances, but also at any distances from the charge q, because at small distances - when the concept of the screening action of a plasma loses its meaning - in the limit formula (4.8) leads to the correct result $\varphi = \frac{q}{r}$. /91

Condition (4.10) is fulfilled to a better extent, the higher is the temperature of the plasma, and the smaller is its density. In all cases which are of physical interest, it is fulfilled very well. For example, even for a dense plasma when $n = 10^{21}$ particles/cm^3, this condition means $T >> 2.3 \cdot 10^{-12}$erg = 1.4 ev.

4. When the electric charge is equally distributed along an infinite plane, equation (4.2) can be readily solved very exactly, without imposing the limitation (4.3). Let us assume that a plasma occupies half space to the right of a uniformly-charged plane AB (Figure 2). Let the charge of the plane be held constant. We can find the electric field in the plasma after the state of static equilibrium is reached.

We shall direct the x-axis clockwise, perpendicularly to the plane AB, placing the origin at the point 0. In the case being considered, all of the quantities depend only on one coordinate x, and equation (4.2) changes into

$$\frac{dE}{dx} = -8\pi en \operatorname{sh}\frac{e\varphi}{T}, \tag{4.11}$$

where $E = -\frac{d\varphi}{dx}$ is the intensity of the electric field. After multiplying equation (4.11) by $2Edx = -2d\varphi$, we obtain

$$dE^2 = 16\ en \operatorname{sh}\frac{e\varphi}{T} d\varphi.$$

The solution of this equation, which becomes zero for $x = \infty$, is

$$E^2 = 16\pi nT\left(\operatorname{ch}\frac{e\varphi}{T} - 1\right) = 32\pi nT \operatorname{sh}^2\frac{e\varphi}{2T}.$$

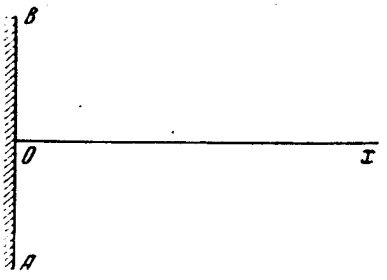

Figure 2.

In the extraction of the square root, the plus sign must be taken. If the plane AB has a positive charge, then the field E is directed to the positive side of the x-axis, i.e., it is positively charged. /92
The potential φ is thereby also positive. If the plane AB is charged negatively, then E and φ are also negative. Only a square root with the plus sign satisfies these conditions. Thus,

$$E = -\frac{d\varphi}{dx} = +\sqrt{32\pi nT} \operatorname{sh}\frac{e\varphi}{2T}. \tag{4.12}$$

After integration, we have

$$\operatorname{th}\frac{e\varphi}{4T} = \operatorname{th}\frac{e\varphi_0}{4T} e^{-\frac{x}{D}}, \tag{4.13}$$

where φ_0 is the potential of the AB plane, and D is the Debye radius determined by expression (4.5). Solving equation (4.13) with respect to φ, we find

$$\varphi = \frac{2T}{e} \ln \frac{1 + \text{th}\,\frac{e\varphi_0}{4T} e^{-\frac{x}{D}}}{1 - \text{th}\,\frac{e\varphi_0}{4T} e^{-\frac{x}{D}}}. \qquad (4.14)$$

We must now find φ_0. In view of formula (4.12), the potential is connected with the intensity of the electric field E_0, close to AB, and with the surface density of the charge σ by the relationship

$$E_0 = 4\pi\sigma = \sqrt{32\pi n T}\ \text{sh}\,\frac{e\varphi_0}{2T},$$

from which it follows that

$$\varphi_0 = \frac{T}{e} \ln \left\{ \frac{\pi\sigma}{\sqrt{2\pi n T}} + \sqrt{\frac{\pi\sigma^2}{2nT} + 1} \right\}. \qquad (4.15)$$

Just as is the case for a point charge, in the case being considered the plasma has a screening action which weakens the electric field of the charged plane. As previously, this weakening action is determined by the value of the Debye radius D. The quantity $\text{th}\,\frac{e\varphi}{4T}$ decreases exponentially with the distance from the surface, while throughout the Debye radius it decreases e times.

For $x >> D$, formula (4.14) changes into

$$\varphi = \frac{4T}{e} \cdot \text{th}\,\frac{e\varphi_0}{4T} \cdot e^{-\frac{x}{D}}. \qquad (4.16)$$

The potential itself φ and the intensity of the electric field corresponding to it decrease exponentially at such distances.

5. In certain cases (see Section 9 and 12) *quasi-stable states* can arise in a plasma, which are characterized by Maxwell distributions of the ion and electron velocities, but with different *ion* and *electron* temperatures. The concept of Debye screening and Debye radius can be extended to these states in a simple way. The difference only consists of the fact that - instead of expression (4.5) - the following expression must be used: /93

$$D = \sqrt{\frac{T_i T_e}{4\pi n e^2 (T_i + T_e)}}, \qquad (4.17)$$

where T_i *is the ion temperature;* T_e *is the electron temperature,* and n is the concentration of either ions or electrons (it is assumed that the plasma is *quasi neutral*).

6. In a calculation of the Coulomb logarithm, the radius of action for the Coulomb forces is usually cut off at D. We shall proceed in the same way, although it should be noted that the justification for this procedure - which employs the Debye screening in

its literal sense - entails confused statements and contradictions. If the Debye screening is taken literally, it should be indicated that each plasma particle is a force center, around which there is a balanced Maxwell-Boltzman distribution of all the other plasma particles. But it is clear that this idea is contradictory, since the distribution of one and the same particles can not be a spherically symmetrical Maxwell-Boltzman distribution with the force center simultaneously at the point where particle *1* is located, and at the point where particle *2* is located. A balanced Maxwell-Boltzman distribution can be established around the force center only when this center is fixed. The plasma particles effect thermal motion. Spherically symmetrical Maxwell-Boltzman distribution cannot be established around each plasma particle. Therefore, Debye screening in its literal sense - when each plasma particle is regarded as a force center of Maxwell-Boltzman distribution - cannot be applied to the plasma.

Nevertheless, in terms of calculations it leads to correct results primarily. The fact is that each charged plasma particle attracts particles which have a different charge and repulses particles having the same charge. Due to this fact, after some short intervals of time, an excess of particles having an opposite charged sign, and a deficiency of particles with the same charged sign, occur around each plasma particle. The distribution of these particles has nothing in common with the balanced Maxwell-Boltzman distribution. It exists for a period of time on the order of the period of plasma (Langmuir) fluctuations, is gradually destroyed by the thermal motion, and then is formed again. On the average, this has the same effect as that which would be produced if the Coulomb field of each plasma particle were cut off at a certain definite distance R. As a more detailed analysis will show (see Section 19), this distance is on the order of the Debye radius D. The probability of this result becomes more apparent, if any plasma particle which has been at rest for a long period of time - for whatever reason - is investigated. The concept of Debye screening can be applied to those particles; in the calculational sense, this is equivalent to cutting off the radius of action Coulomb forces at a point on the order of D. The exact value of the cut off radius R is unimportant, since R always enters into the theory in the *argument of the logarithm* and the error in the magnitude of R, by a considerable factor, barely appears in the values of the Coulomb logarithm. Therefore, in the following Section we shall calculate the Coulomb logarithm on the basis of Debye screening. It is impossible to make such calculations in an absolute manner, but they yield sufficiently accurate values of the Coulomb logarithm. /94

5. CALCULATION OF THE COULOMB LOGARITHM.

1. In order to calculate the Coulomb logarithm, it is necessary to determine the differential cross-section $\sigma(\vartheta, u)$ of a charged

particle in the Debye field of another particle, which is determined by the potential $\varphi = \frac{e^*}{r} e^{-\frac{r}{D}}$. It would also be possible to determine this cross-section in the cut off Coulomb field, i.e., the field whose potential is $\frac{e^*}{r}$ for $r < D$, and which becomes a constant for $r > D$. However, this method entails cumbersom calculations - although they are simple in principle - and there is no justification for dealing with them within the framework of the pair collision approach. Therefore, we shall use another, much simpler method.

2. Let us first examine the Coulomb logarithm calculation in that region where *classical mechanics* is applied. In a classical treatment, the result of the collision of two particles is determined by two parameters - the *relative velocity* u of the colliding particles before collision, and the *impact parameter* ρ between them. Thus, in order to simplify the calculations, we shall not cut off the radius of action of the Coulomb forces, but rather the impact parameter ρ, assuming that the maximum value of ρ equals the Debye radius D. In other words,we shall assume that a test particle with $\rho < D$ is influenced by a field particle not only when it falls within the limits of the Debye sphere of the latter, but also throughout its entire motion. For such particles, the cross-section $\sigma(\vartheta, u)$ is determined by the Rutherford formula (3.5), and for particles with $\rho > D$ this cross-section equals zero. This cut-off method leads to a somewhat exaggerated value for the cross-section σ and, consequently, for the Coulomb logarithm, as compared with the cut-off radius of action of the Coulomb forces. This exaggeration, however, is not important, because the field particle influences the test particle primarily within the limits of the Debye sphere of the former.

As is well-known from the solution of the Kepler problem in classical mechanics, the impact parameter ρ is connected with the angle of deviation ϑ by the relationship /95

$$\operatorname{tg}\frac{\vartheta}{2} = \frac{\rho_\perp}{\rho}, \tag{5.1}$$

where $\rho_\perp$ is the value of the impact parameter, at which the angle of deviation ϑ equals $\frac{\pi}{2}$. It is equal to:

$$\rho_\perp = \left|\frac{ee^*}{\mu u^2}\right|. \tag{5.2}$$

If the impact parameter is cut off at the Debye radius D, then the minimum value of the angle of deviation is determined by the formula

$$\operatorname{tg}\frac{\vartheta_{min}}{2} = \frac{\rho_\perp}{D}$$

or

$$\sin\frac{\vartheta_{min}}{2}=\frac{\varrho_\perp}{\sqrt{D^2+\varrho_\perp^2}}.$$

The substitution of this value in formula (3.8) gives

$$L_{cl}^{*}=\ln\frac{\sqrt{D^2+\varrho_\perp^2}}{\varrho_\perp}\approx\ln\frac{D}{\varrho_\perp}, \tag{5.3}$$

where the index of L indicates that classical mechanics was used in calculating the Coulomb logarithm.

In all of the cases in which we are interested, the Coulomb logarithm is a large quantity (exceeding 10, see Tables 1 and 2). Since the maximum impact parameter of D enters into formulas (5.3) in the argument of the logarithm, then expression (5.3) is not sensitive to changes in D. For example, if a quantity which is twice as large is taken instead of D, then L is increased by $2 \tilde{-} 0.7$ in all. This insensitivity also justifies the substitution of the cut-off radius of action of Coulomb forces by the cut-off impact parameter ρ which was done above to simplify the calculations.

3. Classical mechanics is applicable under the condition $\frac{2\pi}{\lambda}\rho_\perp \gg 1$, where $\lambda=\frac{h}{\mu u}$ is the *de Broglie wave length* for the test particle within the frame of reference in which the dispersion center (field particle) is at rest. Substituting expression (5.2) instead of $\rho_\perp$, we can write this condition in the following form:

$$u \ll \alpha c, \tag{5.4}$$

where

$$\alpha=\left|\frac{ee^*}{\hbar c}\right| \tag{5.5}$$

$\left(\hbar=\frac{h}{2\pi}=1.05\cdot10^{-27}\ \text{erg}\cdot\text{sec}\right)$. If e and e^* equal an elementary charge, then the constant $\alpha=\frac{e^2}{\hbar c}=\frac{1}{137}$ coincides with the *fine structure constant*.

If condition (5.4) is not observed, the classical treatment is not applicable. At first glance, this could seem incomprehensible, because an exact quantum mechanical solution of the problem regarding the scattering of a charged particle in a Coulomb field of dispersion center leads to the expression for $\sigma(\vartheta, u)$ which accurately coincides with the classical expression (3.5) [see, for example, the book (Ref. 3) or any detailed course on quantum mechanics]. The fact is, however, /96

* Note: cl designates 'classical'.

that this agreement occurs when the field of a dispersion center is a Coulomb field *throughout the entire space*. For a *cut-off* Coulomb field, the wave properties of the particles deviate significantly from the properties which a classical investigation yields[1].

In the limiting case, when

$$u \gg \alpha c, \tag{5.6}$$

the quantum mechanical problem regarding scattering can be comparatively simply solved in the *B. approximation*. The solution can be most simply obtained not for a cut-off Coulomb field, but for a Debye field with the potential

$$\varphi = \frac{e^*}{r} e^{-\frac{r}{D}}. \tag{5.7}$$

In this case, when condition (5.6) is observed, quantum mechanics - as is known - leads to the following results:

$$\sigma(\vartheta, u) = \left(\frac{ee^*}{2\mu u^2}\right)^2 \frac{1}{\left(\sin^2\frac{\vartheta}{2} + \varepsilon^2\right)^2}, \tag{5.8}$$

where

$$\varepsilon = \frac{\lambda}{4\pi D} = \frac{\hbar}{2\mu u D} \tag{5.9}$$

[see, for example, the books (Ref. 3 - Ref. 5)].

If expression (5.8) is substituted in formulas (3.3) and (3.4), we then again arrive at formulas (3.6) and (3.7), with the only difference being that, instead of the classical value, it is necessary to use the quantum mechanical value of the Coulomb logarithm

$$L_{qu}^* = \frac{1}{4}\int_0^\pi \frac{\sin^2\frac{\vartheta}{2}\sin\vartheta}{\left(\sin^2\frac{\vartheta}{2} + \varepsilon^2\right)^2} d\vartheta = \frac{1}{2}\ln\frac{1+\varepsilon^2}{\varepsilon^2} - \frac{1}{2(1+\varepsilon^2)}. \tag{5.10}$$

---

[1] However, it should be noted that in quantum mechanics the Rutherford formula (3.5) is strictly applicable only for collisions of *non-identical* particles. In the collisions of *identical* particles (for example, electrons or similar ions) specific *exchange effects* arise which lead to an additional term in the expression for $\sigma(\vartheta, u)$. Therefore, the Rutherford formula (3.5), and also formula (5.8), strictly speaking, are not applicable to collisions of identical particles. We cannot take these exchange effects into consideration, because the scatter is significant only at *small angles*, when these effects are negligible, in the problems in which we are interested.

* Note: *qu* designates 'quantum'.

Since, in all of the cases which are of physical interest, /97

$$\varepsilon \equiv \frac{\lambda}{4\pi D} \ll 1, \tag{5.11}$$

the square of ε can be disregarded as compared with unity. In this approximation

$$L_{q\text{м}} = \ln \frac{1}{\varepsilon} - \frac{1}{2} = \ln \frac{4\pi D}{\lambda} - \frac{1}{2}. \tag{5.12}$$

If the components $-\frac{1}{2}$ are disregarded, this expression differs from the classical value (5.3) only insofar that it contains $\frac{\lambda}{4\pi}$, instead of the lower limit $\rho_\perp$. This result can be readily understood. A de Broglie wave, which is connected with an incident particle, undergoes *diffraction* in the Debye sphere surrounding the dispersion center. It follows from the diffraction theory or from elementary interference considerations that the mean value of the *diffraction angle* equals $\vartheta = \frac{\lambda}{2D}$, correct to an unimportant coefficient on the order of unity. If this value exceeds the classical limit $\vartheta_{\min} = \frac{2\rho_\perp}{D}$, then the classical formula (5.3) becomes inapplicable, and $\vartheta \approx \frac{\lambda}{2D}$ must be taken as the lower limit in the integral (3.8). This leads to $L = \ln \frac{4D}{\lambda}$, which differs from formula (5.12) by the unimportant multiplier π in the argument of the logarithm. It also follows from this qualitative investigation that the result of scattering in the cut-off Coulomb field cannot significantly differ from the result of scattering in the Debye field (5.7).

The quantum formula (5.12) can be written in the form

$$L_{q\text{м}} = L_{\text{кл}} + \ln \frac{2\alpha c}{u} - \frac{1}{2}. \tag{5.13}$$

We should note that this formula is derived on the assumption that $u \gg \alpha c$ since the classical expression (5.3) is suitable for $u \ll \alpha c$.

4. In the intermediate region $u \approx \alpha c$, the quantum mechanical calculations become too complex. There is no physical justification for dealing with them, since - within the framework of the pair collision theory with its artificial and, to a considerable degree, arbitrary cut-off radius of action of the Coulomb forces - the refinement of the Coulomb logarithm values, which are obtained as a result of these complex calculations, is illusory. It is much simpler and more consistent, within the framework of the given approach, to pursue the following method. As both limiting formulas (5.3) and (5.12) show,

the Coulomb logarithm contains the velocity u in the argument of the logarithm, and, consequently, it is a very slowly changing function of u. It is physically apparent that such a slow change remains in the intermediate region. Therefore, if no significant errors are introduced, it is possible to extrapolate the expressions (5.3) and (5.12) in the intermediate region up to a value of $u = u_{bn}$*, at which these expressions coincide. For $u < u_{bn}$, the classical formula (5.3) should be used, and for $u > u_{bn}$ - the quantum formula (5.12) or (5.13). /98

As can be seen from formula (5.13), the value of u_{bn} is determined from the requirement $\ln \frac{2\alpha c}{u_{bn}} = \frac{1}{2}$, which gives

$$u_{bn} = 1{,}21\alpha c = 0{,}00885c, \tag{5.14}$$

while the numerical coefficient 0.00885 refers to the collision of singly-charged ions.

If the *equivalent temperature* with respect to the formula $3T = \mu u^2$ is introduced, instead of the relative velocity u, then equality (5.14) changes to

$$T_{bn} = 0{,}49\mu\alpha^2c^2. \tag{5.15}$$

Substituting the corresponding values of reduced mass, in the case of a deuterium plasma we obtain the following values of the *boundary temperatures* for electron-electron, electron-ion, and ion-ion collisions, respectively:

$$\left.\begin{aligned} T_{bn}^{ee} &= 6{,}65 \text{ ev} \\ T_{bn}^{ei} &= 13{,}3 \text{ ev} \\ T_{bn}^{ii} &= 2{,}45\cdot 10^4 \text{ ev} = 24{,}5 \text{ Kev}. \end{aligned}\right\} \tag{5.16}$$

5. In the case when any plasma particle plays the role of a test particle, when one calculates the Coulomb logarithm, it is possible to disregard the scattering of the velocities, taking a certain mean velocity as u, due to the slight dependence of the Coulomb logarithm on u. If the electron and ion temperatures do not differ too greatly from each other, then it is possible to introduce a mean temperature for them of T, determining u by the equality $\mu u^2 = 3T$. We can substitute the quantity T, determined in this way, in the expression (4.5) for the Debye radius D, after which we can find the value of the Coulomb logarithm from the

* Note: bn designates 'boundary'.

classical formula (5.3)

$$L_{cl} = 23{,}1 - \frac{1}{2}\ln n + \frac{3}{2}\ln T_{ev}, \tag{5.17}$$

where the temperature is given in electron volts. This expression can be used for collisions of any type of singly-charged particles, i.e., electron-electron, ion-ion, electron-ion collisions. However, its limit of applicability depends on the mass of the colliding particles. It is applicable for $T \ll T_{bn}$, where T_{bn} is determined by the values of (5.16).

For $T > T_{bn}$, it is necessary to introduce a quantum correction with respect to the formula (5.13) in the expression (5.17). After this we obtain: /99

$$\left.\begin{aligned}
&1)\ \text{for electron-electron collisions}\\
&L_{qu} = 24{,}1 - \frac{1}{2}\ln n + \ln T_{ev}\ (T_{ev} \geqslant 6{,}65\,ev);\\
&2)\ \text{for electron-ion collisions}\\
&L_{qu} = 24{,}4 - \frac{1}{2}\ln n + \ln T_{ev}\ (T_{ev} \geqslant 13{,}3\,ev);\\
&3)\ \text{for ion-ion collisions (deuterium)}\\
&L_{qu} = 28{,}2 - \frac{1}{2}\ln n + \ln T_{ev}\ (T_{ev} \geqslant 2{,}45\cdot 10^4\,ev).
\end{aligned}\right\} \tag{5.18}$$

The classical (5.17) and the quantum (5.18) expressions lead to one and the same dependence of the Coulomb logarithm L on the concentration of electrons n, but give a different dependence on temperature T. Figure 3 schematically shows the dependence of L on $\ln T$ at a constant concentration of n. The rectilinear sections AB and CD correspond to the regions of applicability for the classical (5.17) and quantum (5.18) expressions. In the intermediate region, the dependence would have to be expressed by the curve BC of the hyperbolic type, for which the lines AB and CD are asymptotes. However, we continued the lines AB and CD up until they intersected at the point E, and we replaced the curved section BC by the broken line BEC, overestimating the values of the Coulomb logarithm a little.

As can be seen from expressions (5.18), for electron-electron collisions the values of L barely differ from the corresponding values for electron-ion collisions, and therefore this difference can be disregarded. Table 1 lists the values of the Coulomb logarithms L for electron-ion collisions, and Table 2 - those for ion-ion collisions. It is assumed that the ion and electron temperatures are the same. The values of L for an electron-proton plasma, which are given in a book by Spittser (Ref. 6), barely differ from the values given by /100

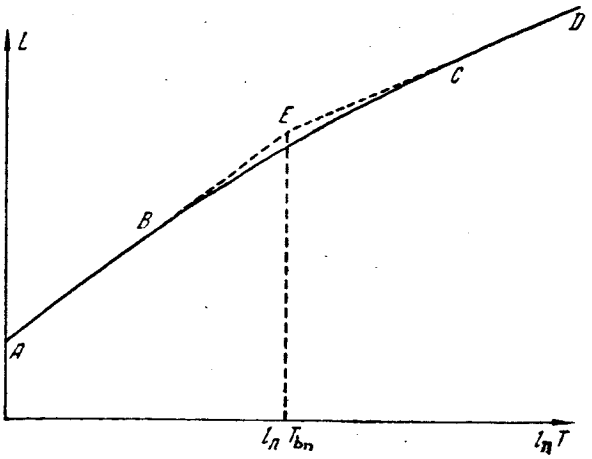

Figure 3.

us in Table 1. From these tables, and also from an analysis of the formulas with which they were obtained, it can be seen that the difference in the magnitude of L for electron-ion and ion-ion collisions is small, as are the quantum corrections, and in the majority of cases they can be disregarded.

TABLE 1

VALUES OF L FOR ELECTRON-ION COLLISIONS

$T_{e\omega}$	Electron Concentration n, cm^{-3}							
	1	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}	10^{21}
10^{-1}	19,7	16,2	12,8	9,3	5,9			
1	23,1	19,7	16,2	12,8	9,3	5,9		
10	26,6	23,1	19,7	16,2	12,8	9,3	5,9	
10^2	29,0	25,6	22,1	18,7	15,2	11,8	8,3	4,9
10^3	31,3	27,9	24,4	21,0	17,5	14,1	10,6	7,2
10^4	33,6	30,2	26,7	23,3	19,8	16,4	12,9	9,5
10^5	35,9	32,5	29,0	25,6	22,1	18,7	15,2	11,8
10^6	38 2	34,8	31,3	27,9	24,4	21,0	17,5	14,1

6. ENERGY EXCHANGE BETWEEN A TEST PARTICLE AND A PLASMA. GENERAL FORMULAS

1. Let us again turn to one of the problems which was examined in Section 3. Let us assume that a test particle moves in a plasma with a mass m, a charge e, and a velocity v. It is necessary to find the mean rate of its kinetic energy change $\mathcal{E}$. If we disregard

the dependence of the Coulomb logarithm L on the relative velocity u, then the problem is reduced to calculating the analog of the electric field E_v, and its potential φ_v in velocity space. /101

TABLE 2

VALUES OF L FOR ION-ION COLLISIONS (DEUTERIUM)

T_{ev}	Electron Concentration n_e, cm^{-3}							
	1	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}	10^{21}
10^{-1}	19,7	16,2	12,8	9,3	5,9			
1	23,1	19,7	16,2	12,8	9,3	5,9		
10	26,6	23,1	19,7	16,2	12,8	9,3	5,9	
10^2	30,0	26,6	23,1	19,7	16,2	12,8	9,3	5,9
10^3	33,5	30,0	26,6	23,1	19,7	16,2	12,8	9,3
10^4	36,9	33,5	30,0	26,6	23,1	19,7	16,2	12,8
10^5	39,7	36,2	32,8	29,3	25,9	22,4	19,0	15,5
10^6	42,0	38,5	35,1	31,6	28,2	27,4	21,3	17,8

Let us assume that the distribution function $f^*(\mathbf{v}^*) \equiv f^*(v^*)$ of the field particles in velocity space is *isotropic*, i.e., it depends only on the absolute value of the velocity v^*, and not on its direction. Then the quantity ρ_v, which is determined by expression (3.15), will also depend only on v^*. The problem is reduced to calculating the electrostatic field for spherically symmetrical distribution of the electric charges around the origin. This problem can be solved in an elementary way with the aid of the *Gauss electrostatic theorem*. The latter gives

$$\left.\begin{aligned} \mathbf{E}_v(\mathbf{v}) &= \frac{\mathbf{v}}{v^3}\int\limits_{v^*<v} \varrho_v d\mathbf{v}^*, \\ \varphi_v(\mathbf{v}) &= \frac{1}{v}\int\limits_{v^*<v} \varrho_v d\mathbf{v}^* + \int\limits_{v^*>v} \frac{\varrho_v}{v^*} d\mathbf{v}^*. \end{aligned}\right\} \tag{6.1}$$

Let us carry out the calculation for the Maxwell velocity distribution of field particles.

$$f^*(\mathbf{v}^*) = n^*\left(\frac{b^*}{\sqrt{\pi}}\right)^3 e^{-b^{*2}v^{*2}} \qquad \left(b^* = \sqrt{\frac{m^*}{2T^*}}\right). \tag{6.2}$$

Here T^* is the temperature of a definite type of field particle. Due to the insignificant difference in their mass, the temperature T^* is the same for all of the ions. But for the electrons - in view of the fact that their mass is small - the temperature T^* can differ from the ion temperature (see Section 9 and 12).

Substituting expression (3.15) in formula (6.1), instead of ρ_v, and taking the volume of the spherical layer $dv^* = 4\pi v^{*2} dv^*$ as the volume element, as a result of simple integration we obtain:

$$\mathbf{E}_v(\mathbf{v}) = \frac{4\pi (ee^*)^2 n^*}{v^3} \Phi_1(b^* v) \cdot \mathbf{v}, \tag{6.3}$$

$$\varphi_v(\mathbf{v}) = \frac{4\pi (ee^*)^2 n^*}{v} \Phi(b^* v). \tag{6.4}$$

The *error integral* is designated by $\Phi(x)$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi. \tag{6.5}$$

The function $\Phi_1(x)$ is determined by the expression

$$\Phi_1(x) = \Phi(x) - \frac{2x}{\sqrt{\pi}} e^{-x^2} = \Phi(x) - x \frac{d\Phi}{dx}. \tag{6.6}$$

Substituting expressions (6.3) and (6.4) in formula (3.11), /102
we obtain

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = -\frac{4\pi e^2}{v} \sum^* Ln^* e^{*2} \left[\frac{\Phi(b^* v)}{m^*} - \frac{2b^* v (m + m^*)}{mm^* \sqrt{\pi}} e^{-b^{*2} v^2} \right]. \tag{6.7}$$

This formula determines the mean rate of kinetic energy change $\mathcal{E}$ of a test particle moving in a plasma with a definite velocity v. We should note that quantity $\left\langle \frac{d\mathcal{E}}{dt} \right\rangle$ is obtained from $\frac{d\mathcal{E}}{dt}$ by averaging with respect to a group of test particles, which is characterized by one and the same velocity values v. Such a group can be represented, for example, by a bundle of identical test particles which do not interact with each other and which move in a plasma at the same velocity v. (The velocity directions for the different particles in the bundle cannot coincide, but the absolute velocity values must be the same).

2. The functions $\Phi(x)$ and $\Phi_1(x)$ will be constantly encountered as we proceed. The values of $\Phi(x)$ and $\frac{d\Phi}{dx}$ are given in the "Tables of Functions" of Yanke and Emde (Ref. 7), with an accuracy which is sufficient for our purposes. For small x, it is convenient to use the expansion

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\, n!}. \qquad (6.8)$$

It is obtained from formula (6.5) by expanding the integrand in a power series with subsequent integration of it term by term. Utilizing formula (6.6) and expansion (6.8), we obtain

$$\Phi_1(x) = \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+3)\, n!}. \qquad (6.9)$$

For large x, it is more convenient to use *asymptotic series*. Representing $\Phi(x)$ in the form

$$\Phi(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\xi^2} d\xi,$$

we can readily obtain the following with integration by parts

$$\Phi(x) = 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \left[\frac{1}{x} + \sum_{k=1}^{n} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{2^k \cdot x^{2k+1}} \right] + R_n(x),$$

Where "the residual term" $R_n(x)$ is given by the expression /103

$$R_n(x) = \frac{(-1)^{n+1}}{\sqrt{\pi}} \int_x^{\infty} \frac{1 \cdot 3 \cdots (2n+1)}{2^{n+1}} e^{-\xi^2} d \frac{1}{\xi^{2n+3}}.$$

Thus,

$$|R_n(x)| < \frac{e^{-x^2}}{\sqrt{\pi}} \cdot \frac{1 \cdot 3 \cdots (2n+1)}{2^{n+1} \cdot x^{2n+3}}.$$

Thus, if function $\Phi(x)$ is in agreement with the asymptotic theories

$$\Phi(x) \to 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \left[\frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{2^k \cdot x^{2k+1}} \right] \qquad (6.10)$$

and if it is truncated at any point, then this truncated series will approximate the function $\Phi(x)$ with an error whose absolute magnitude does not exceed the first discarded term. The sign of the error agrees with the sign of this term.

In a similar way,

$$\Phi_1(x) \to 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \left[2x + \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{2^k \cdot x^{2k+1}} \right]. \qquad (6.11)$$

7. CRITICAL VELOCITY AND MAXIMUM ENERGY TRANSFER

1. For a detailed study of energy exchange between a test particle and a plasma, let us examine an individual component in the right-hand side of formula (6.7). In physical terms this designates the *mean velocity* of kinetic energy increase for a test particle, due to its interaction with one separate component of the plasma - for example, with all of the electrons or with a definite type of ion. This component can be designated by $\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_1$. The index "1" must indicate that we are dealing with the interaction of a test particle, not with the entire plasma, but only with one imaginary separate component. According to formula (6.7),

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_1 = -\frac{4\pi (ee^*)^2}{v} Ln^* \left[\frac{\Phi (b^* v)}{m^*} - \frac{2b^* v (m + m^*)}{mm^* \sqrt{\pi}} e^{-b^{*2} v^2} \right]. \qquad (7.1)$$

This separation, of the interaction of a test particle with a plasma, into interaction with separate components has a conditional-mathematical - nature. We are not dealing with the interaction of a test particle with any isolated component in the plasma, but rather with the interaction in the presence of all of the other plasma components. The role of the latter can be reduced to the compensation of the Coulomb repulsive forces between the particles of an isolated plasma component, and only if they are present is there any point in speaking about Coulomb screening and Coulomb logarithm. /104

We have this conditional meaning in mind, when we speak about the interaction of a bundle of non-interacting particles with an isolated plasma component.

2. Let us introduce the notation

$$x = b^* v, \quad \beta = \frac{m^*}{m}; \qquad (7.2)$$

$$F(x, \beta) = \Phi (x) - \frac{2x}{\sqrt{\pi}} (1 + \beta) e^{-x^2}. \qquad (7.3)$$

The parameter x represents the relationship of the velocity of a test particle to the *most probable velocity* $\frac{1}{b^*} = \sqrt{\frac{2T^*}{m^*}}$ of particles in an isolated plasma component. In these designations:

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_1 = -\frac{4\pi}{m^*} Lb^* (ee^*)^2 n^* \frac{F(x, \beta)}{x}. \qquad (7.4)$$

Let us determine at what values of x this expression becomes zero. In order to do this, it is necessary that $F(x, \beta) = 0$. The function $F(x, \beta)$ becomes zero for $x = 0$. However, this root must be discarded, since for $x = 0$ the relationship $\frac{F(x, \beta)}{x}$ does not

become zero, but equals $\frac{2\beta}{\sqrt{\pi}}$. The derivative

$$\frac{\partial F}{\partial x} = \frac{2}{\sqrt{\pi}} e^{-x^2} [2x^2 (1 + \beta) - \beta]$$

is negative for $x^2 < \frac{\beta}{2(1 + \beta)}$ and positive for $x^2 > \frac{\beta}{2(1 + \beta)}$. At the point $x = \sqrt{\frac{\beta}{2(1 + \beta)}}$ the quantity F as a function of x reaches the minimum, remaining negative in the interval $0 < x < \sqrt{\frac{\beta}{2(1 + \beta)}}$. From this point on, it increases monotinically and strives to 1 for $x \to \infty$ (Figure 4). Therefore, in the interval $\sqrt{\frac{\beta}{2(1 + \beta)}} < x < + \infty$, the function $F(x, \beta)$ becomes zero only once. Consequently, the equation $\frac{F(x, \beta)}{x} = 0$ has a single root $x = x_{root}$, and this root lies in the same interval. The value of this root is determined by the equation $F(x_{root}, \beta) = 0$, which can be written in the form

$$1 + \beta = \frac{\sqrt{\pi}}{2x_{root}} \Phi(x_{root}) e^{x^2_{root}}. \qquad (7.5)$$

The quantity x_{root}, and the velocity $v_{root} = \frac{x_{root}}{b^*} = x_{root} \sqrt{\frac{2T^*}{m^*}}$ /105

which corresponds to it, will be called *critical*. At the critical velocity of a test particle, the mean rate of energy transfer in an isolated plasma component becomes zero. If a bundle of non-interacting test particles is introduced, as in Section 6, then it can be stated that, for $v < v_{root}$, the energy is transmitted to the bundle from the isolated plasma component, and for $v > v_{root}$ - from the bundle to the same plasma component.

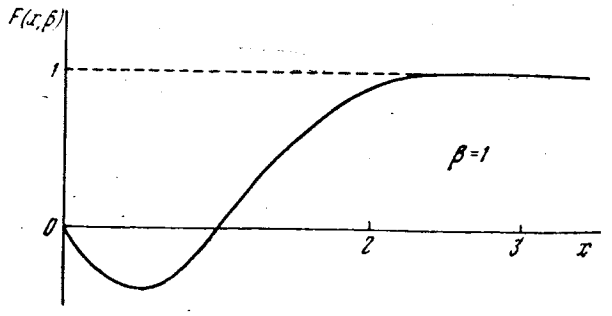

Figure 4

The maximum rate of energy transfer from the bundle to an isolated plasma component takes place for $x = x_m$, i.e., $v = v_m = \frac{x_m}{b^*}$. The value of x_m is determined by the equation $\frac{\partial}{\partial x} \frac{F(x, \beta)}{x} = 0$, which can

be written in the form

$$1 + 2x_m^2 (1 + \beta) = \frac{\sqrt{\pi}}{2x_m} \Phi(x_m) e^{x_m^2}. \tag{7.6}$$

This equation has two roots. The first root $x = 0$, which is of little interest, corresponds to the case of a fixed test particle (the energy is transmitted from the plasma to the test particle). The existence of the second root can be proven with the aid of the following simple remarks. The function $\frac{F(x, \beta)}{x}$ becomes zero for $x = x_{root}$ and $x = \infty$. Therefore, according to the well-known *Rolle theorem*, there must be a value of $x = x_m$, which lies in the interval $x_{root} < x_m < \infty$, for which the derivative $\frac{\partial}{\partial x} \frac{F(x, \beta)}{x}$ becomes zero. In this case, the energy passes from the bundle of test particles to an isolated plasma component, since $x_m > x_{root}$. From this point on x_m will designate the value of the second root. It can be shown that equation (7.6) does not have other roots (we shall not deal with this problem).

The roots x_{root} and x_m increase with an increase in β. This can be readily verified, by calculating the following derivatives with the aid of relationships (7.5) and (7.6): /106

$$\frac{dx_{root}}{d\beta} = \frac{x_{root}}{2x_{root}^2 (1 + \beta) - \beta}; \tag{7.7}$$

$$\frac{dx_m}{d\beta} = \frac{x_m}{1 + (1 + \beta)(2x_m^2 - 3)}. \tag{7.8}$$

The first of these derivatives is positive, because $x_{root} > \sqrt{\frac{\beta}{2(1 + \beta)}}$, as was shown above. The second derivative (7.8) is also positive, as it will be shown below that $2x_m^2 > 3$.

3. In investigating equations (7.5) and (7.6), we shall consider three cases.

First Case. The parameter $\beta = \frac{m^*}{m}$ is very small. This case is realized, for example, for a bundle of ions which interacts with electrons in a plasma. The value of $x = \sqrt{\frac{\beta}{2(1 + \beta)}}$, at which the function $F(x, \beta)$ has a minimum, strives to zero for $\beta \to 0$. Therefore, it is natural to expect that for $\beta \to 0$ the root of equation (7.5) also strives to zero, and, consequently, this root is very small in the case under consideration. Therefore, it is possible to expand the right part of equation (7.5) in powers of x_{root} and to truncate this expansion, for example, at the terms of the sixth power. As a result, we obtain

$$\frac{3}{2}\beta = x_{root}^2 + \frac{2}{5}x_{root}^4 + \frac{8}{105}x_{root}^6.$$

By squaring both parts of this equality first, and then cubing them, we obtain

$$\frac{9}{4}\beta^2 = x_{root}^4 + \frac{4}{5}x_{root}^6,$$

$$\frac{27}{8}\beta^3 = x_{root}^6.$$

Eliminating x_{root}^4 and x_{root}^6, we find

$$x_{root}^2 = \frac{3}{2}\beta - \frac{9}{10}\beta^2 + \frac{243}{350}\beta^3. \tag{7.9}$$

We can thus obtain higher powers in the expansion of x_{root}^2 in powers of β. For small β, we can confine ourselves to the first term

$$x_{root}^2 = \frac{3}{2}\beta. \tag{7.10}$$

The root x_{root} was small, which confirms the applicability of the expansion which we used. /107

Let us now find the expression for x_m. For $\beta = 0$, the root of transcendental equation (7.6) is approximately

$$x_{m0} \approx 1{,}52. \tag{7.11}$$

Utilizing this result, we can readily find the root x_m for $\beta \neq 0$ by the method of successive approximations. It is simpler not to start with equation (7.6), but to use the relationship (7.8) which is obtained from it. Expanding the right part of this equation in powers of β, we obtain

$$\frac{dx_m}{d\beta} = \frac{x_m}{2(x_m^2-1)}\left[1 - \frac{2x_m^2-3}{x_m^2-1}\frac{\beta}{2} + \left(\frac{2x_m^2-3}{x_m^2-1}\cdot\frac{\beta}{2}\right)^2 - \cdots\right].$$

In the first approximation let us retain only the first term in the right part, substituting x_m by x_{m0} in it. As a result, after integration, we obtain

$$x_m = x_{m0} + \frac{x_{m0}}{2(x_{m0}^2-1)}\beta \approx 1{,}52 + 0{,}580\beta. \tag{7.12}$$

The following approximations are obtained in a similar way. In the third approximation

$$x_m \approx 1{,}52 + 0{,}580\beta - 0{,}458\beta^2 + 0{,}57\beta^3. \tag{7.13}$$

Second Case. The parameter $\beta \equiv \frac{m^*}{m}$ is very large. A bundle of electrons interacting with plasma ions can serve as an example. For hydrogen ions, $\beta = 1836$, and for deuterium ions, $\beta = 3672$. In the case under consideration, the roots x_{root} and x_m are comparatively large, and therefore it can be assumed that $\Phi(x_{root}) = \Phi(x_m) = 1$. For $\beta > 15$, the error which is thus introduced does not exceed 1%, and for $\beta > 100$, 0.1%. In this approximation, equations (7.5) and (7.6) change to

$$Ax_{root}e^{-x_{root}^2} = 1, \tag{7.14}$$

$$2Ax_m^3 e^{-x_m^2} = 1, \tag{7.15}$$

while in the second of these equations we have disregarded terms on the order of one, as compared with $2x_m^2(1 + \beta)$. The constant

$$A = \frac{2}{\sqrt{\pi}}(1 + \beta). \tag{7.16}$$

is designated by A. After taking the logarithm, we have

$$x_{root}^2 = \ln A + \ln x_{root},$$
$$x_m^2 = \ln(2A) + 3\ln x_m.$$

Since the coefficient A is large, these equations can be conveniently solved by the method of successive approximations. In the zero approximation, we discard $\ln x_{root}$ and $3 \ln x_m$, and we find /108

$$\left.\begin{aligned} x_{root}^2 &= \ln A; \\ x_m^2 &= \ln(2A), \end{aligned}\right\} \tag{7.17}$$

In the first approximation, we replace x_{root} and x_m in the argument of the logarithm by their zero-order approximations $\sqrt{\ln A}$ and $\sqrt{\ln(2A)}$, which gives

$$\left.\begin{aligned} x_{root}^2 &= \ln A + \frac{1}{2}\ln\ln A, \\ x_m^2 &= \ln(2A) + \frac{3}{2}\ln\ln(2A). \end{aligned}\right\} \tag{7.18}$$

In a similar manner, in the second approximation

$$\left.\begin{aligned} x_{root}^2 &= \ln A + \tfrac{1}{2}\ln\left(\ln A + \tfrac{1}{2}\ln\ln A\right), \\ x_m^2 &= \ln(2A) + \tfrac{3}{2}\ln\left[\ln(2A) + \tfrac{3}{2}\ln\ln(2A)\right] \end{aligned}\right\} \qquad (7.19)$$

etc.

For $\beta > 100$, the error of the second approximation does not exceed 1%.

For electrons which interact with the proton component in a plasma ($\beta = 1836$), we find $x_{root} = 2.95$, $x_m = 3.47$ by the method indicated, and for electrons interacting with the deuteron component ($\beta = 3672$), $x_{root} = 3.07$, $x_m = 3.58$.

Third Case. The parameter $\beta \equiv \frac{m^*}{m}$ is on the order of unity, or differs from it by several factors of ten. This case is realized for a bundle of electrons which interacts with the electron plasma component, and also for a bundle of ions which interacts with the ion plasma component. In this intermediate region, it is difficult to find simpler approximating expressions for the roots $x_{root}(\beta)$ and $x_m(\beta)$, and it is necessary to solve the transcendental equations (7.5) and (7.6). On the other hand, these roots can be determined most simply by numerical integration of equations (7.7) and (7.8), utilizing the approximate formulas (7.9) and (7.13) .

Table 3 lists the values of the roots $x_{root}(\beta)$ and $x_m(\beta)$ for certain values of the parameter β . It also lists the relationships of the energies $\mathcal{E}_{root} \equiv \frac{mv_{root}^2}{2}$ and $\mathcal{E}_m \equiv \frac{mv_m^2}{2}$ to the temperature T^* of a separate plasma component. The energies $\mathcal{E}_{root}$ and $\mathcal{E}_m$ are calculated by the formulas

$$\left.\begin{aligned} \mathcal{E}_{root} &= \frac{x_{root}^2}{\beta} T^*, \\ \mathcal{E}_m &= \frac{x_m^2}{\beta} T^*. \end{aligned}\right\} \qquad (7.20)$$

4. On first glance, it seems paradoxical that $\mathcal{E}_{root} \neq \frac{3}{2} T^*$. /109
Only in the limiting case of very small values of the parameter $\beta = \frac{m^*}{m}$ (monoenergetic bundle of ions which interacts with plasma electrons) does $\mathcal{E}_{root}$ approach $\frac{3}{2} T^*$. In all the other cases $\mathcal{E}_{root} < \frac{3}{2} T^*$. $\mathcal{E}_{root}$ differs from $\frac{3}{2} T^*$ very greatly for large values of β. Thus, for a bundle of electrons which interact with hydrogen ions in a plasma

TABLE 3

CRITICAL PARAMETERS FOR DIFFERENT VALUES OF THE QUANTITY β

β	x_{root}	x_m	$\frac{\mathcal{E}_{root}}{T^*}$	$\frac{\mathcal{E}_m}{T^*}$	β	x_{root}	x_m	$\frac{\mathcal{E}_{root}}{T^*}$	$\frac{\mathcal{E}_m}{T^*}$
0	0	1,52	1,5	∞	30	2,07	2,68	0,144	0,240
0,1	0,377	1,57	1,42	24,6	40	2,15	2,75	0,115	0,189
0,2	0,519	1,61	1,35	13,0	50	2,21	2,80	0,0980	0,157
0,3	0,620	1,65	1,28	9,10	60	2,25	2,84	0,0840	0,134
0,4	0,699	1,69	1,22	7,13	70	2,28	2,87	0,0743	0,118
0,5	0,765	1,72	1,17	5,92	80	2,31	2,90	0,0670	0,105
0,6	0,822	1,75	1,12	5,10	90	2,34	2,92	0,0612	0,0950
0,7	0,871	1,78	1,08	4,53	100	2,37	2,94	0,0556	0,0867
0,8	0,915	1,81	1,05	4,10	200	2,52	3,08	0,0317	0,0478
0,9	0,954	1,83	1,01	3,73	300	2,60	3,15	0,0226	0,0332
1,0	0,990	1,85	0,980	3,42	400	2,66	3,21	0,0177	0,0258
1,1	1,02	1,87	0,948	3,18	500	2,71	3,25	0,0147	0,0211
1,2	1,05	1,89	0,918	2,99	600	2,74	3,28	0,0125	0,0180
1,3	1,08	1,91	0,897	2,80	700	2,77	3,31	0,0110	0,0157
1,4	1,11	1,93	0,873	2,65	800	2,80	3,33	0,00980	0,0139
1,5	1,13	1,94	0,852	2,51	900	2,82	3,35	0,00887	0,0125
2	1,23	2,01	0,757	2,03	1 000	2,84	3,37	0,00807	0,0114
3	1,37	2,11	0,626	1,49	2 000	2,97	3,49	0,00441	0,00608
4	1,47	2,18	0,541	1,19	3 000	3,04	3,55	0,00308	0,00421
5	1,54	2,24	0,475	1,01	4 000	3,09	3,60	0,00239	0,00322
6	1,60	2,29	0,427	0,875	5 000	3,13	3,63	0,00196	0,00264
7	1,65	2,33	0,388	0,775	6 000	3,16	3,66	0,00165	0,00223
8	1,69	2,36	0,357	0,697	7 000	3,18	3,68	0,00145	0,00194
9	1,73	2,39	0,331	0,635	8 000	3,20	3,70	0,00128	0,00172
10	1,76	2,42	0,310	0,585	9 000	3,22	3,72	0,00116	0,00154
15	1,88	2,52	0,235	0,423	10 000	3,24	3,74	0,00105	0,00140
20	1,96	2,59	0,193	0,335					

($\beta = 1836$), $\mathcal{E}_{root} = \frac{1}{211} T^*$, and for a bundle which interacts with deuterium ions ($\beta = 3672$), $\mathcal{E}_{root} = \frac{1}{388} T^*$.

In actuality, there are no contradictions here. The mean rate of energy transfer would have to become zero for $\langle \mathcal{E} \rangle = \frac{3}{2} T^*$, if the velocity of the bundle particles and the isolated plasma component had a Maxwell distribution with one and the same temperature. Using direct calculations in Section 9, we shall verify the fact that expression (7.1) satisfies this requirement. If there is no Maxwell distribution (for example, if a bundle of particles is monoenergetic), then for $\langle \mathcal{E} \rangle = \frac{3}{2} T^*$ the mean rate of energy transfer must not necessarily become zero. /110

However, the following problem then arises. Let us assume that the energy $\mathcal{E}$ of a particle in a monoenergetic bundle is such that $\mathcal{E}_{root} < \mathcal{E} < \frac{3}{2} T^*$. On the one hand, the energy of the bundle will be transferred to the isolated plasma component, i.e., it begins to decrease. On the other hand, statistical equilibrium must be established, in which the bundle particles have a Maxwell distribution with the mean energy $\langle \mathcal{E} \rangle = \frac{3}{2} T^*$. Ultimately, the energy of the bundle must increase. There are no contradictions here. At first, when the energy of the bundle particles are the same or almost the same, energy is actually transferred from the bundle to the isolated plasma component. From this point on, the interaction of the bundle with the plasma leads to a re-distribution of the velocities, and, beginning at a certain time, the quantity $\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_1$ changes its sign. Ultimately, an equilibrium state is established, in which the mean energies of the bundle particles in the plasma are identical.

Let us assume, for example, that at the initial moment the velocity of all the particles in a monoenergetic bundle equals the critical velocity. Then, at this moment, the mean rate of energy transfer will equal zero. Due to the interaction with plasma particles, the velocity distribution of the bundle particles begins to change. Particles with an energy which is less than the critical energy (we shall call them slow particles) appear in the bundle, along with particles having an energy greater than the critical energy (we shall call them rapid particles). Since the mean rate of energy transfer at the initial moment equals zero, then the energy of the bundle as a whole barely changes at the beginning. Then the slow particles begin to receive energy from the isolated plasma component, and the rapid particles - to deliver energy to it. Since the velocity v is in the denominator expression (7.1), the slow particles receive more energy than the rapid particles deliver during the same period of time. On the whole, the energy of the bundle begins to increase monotonically, and this increase will continue until the velocity distribution in the bundle changes into the Maxwell distribution, with a temperature which equals the plasma temperature.

The monotonic increase of the bundle energy will be observed when the energy of a bundle particle is less that the critical energy at the beginning. If $\mathcal{E} > \mathcal{E}_{root}$, then at the beginning the energy will always be transferred from the bundle to the isolated plasma component. For values of $\mathcal{E}$ which are not very large, the rate of energy transfer can become zero, and then the sign can change. For rather large $\mathcal{E}$, the energy of the bundle will decrease monotonically, and the mean rate of energy transfer, which does not change its sign, /111

strives to zero asymptotically.

8. RELATIVE ROLE OF ION AND ELECTRON COMPONENTS IN PLASMA ENERGY EXCHANGE WITH A MONOENERGETIC BUNDLE OF NON-INTERACTING PARTICLES.

1. Let us assume that a *two-component* plasma consists of electrons and identical ions. We shall assume that the electron and ion temperatures are the same, and we shall designate this by T^*. A bundle of test particles is assumed to be *monoenergetic*, and can be composed either of electrons, or of identical ions. The velocities of energy change $\mathcal{E}$ of a bundle particle, due to its interaction with ion and electron components in a plasma, will be designated by $\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{i^*}$ and $\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{e}$, respectively. For these quantities, formulas (7.4) and (7.3) give

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{i^*} = -\frac{4\pi}{vm_i^*} L (ee_i^*)^2 n_i^* \left[\Phi(x) - \left(1 + \frac{m_i^*}{m}\right) x \frac{d\Phi}{dx} \right]; \tag{8.1}$$

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{e} = -\frac{4\pi}{vm_e} L (ee_e)^2 n_e^* \left[\Phi(y) - \left(1 + \frac{m_e}{m}\right) y \frac{d\Phi}{dy} \right]. \tag{8.2}$$

The following definitions are introduced here:

$$x = b_i^* v = \sqrt{\frac{m_i^*}{m} \frac{\mathcal{E}}{T^*}},$$

$$y = b_e^* v = \sqrt{\frac{m_e}{m} \frac{\mathcal{E}}{T^*}} = x \sqrt{\frac{m_e}{m_i^*}}. \tag{8.3}$$

All of the quantities without an asterisk sign refer to the test particle: m, e, $\mathcal{E}$, v, - represent the mass, charge, energy, and velocity of this particle. The asterisk sign is used to indicate the quantities refering to field particles (the mass m_e and the charge e_e of an electron represent an exception to this). For example, m_i^*, e_i^*, n_i^* designate the mass, charge, and concentration of the field ions.

From this point on, it will be assumed that the velocity of bundle particles exceeds the critical velocity.

2. Let us first consider the case when a bundle consists of ions.

If the energy $\mathcal{E}$ is so great that $y >> 1$ (and, consequently, $x >> 1$), then the expressions in the square brackets in formulas

(8.1) and (8.2) can be assumed to equal unity. In this case, if the quasi neutrality $|n_i^* e_i^*| = |n_e^* e_e|$ is taken into consideration, formulas (8.1) and (8.2) give

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_e : \left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{i^*} = \left| \frac{m_i^* e_e}{m_e e_i^*} \right|, \tag{8.4}$$

i.e., *the ion bundle heats the plasma electrons to a greater extent than it does the ions by approximately a factor of* $\left| \frac{m_i^* e}{m e_i^*} \right|$. *Although the bundle is not slowed down, almost all of its energy goes into heating the electrons, and not the ions.* /112

With a decrease in $\mathcal{E}$, *the ions begin to be heated.* For small y, the right part of formula (8.2) can be expanded in a series in powers of y, and this expansion can be truncated at third power terms. Employing the expansion (6.8), we can readily obtain the following:

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_e : \left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{i^*} = \frac{2}{\sqrt{\pi}} \left| \frac{m_i^* e_e}{m_e e_i^*} \right| y \frac{\frac{2}{3} y^2 - \frac{m_e}{m}}{\Phi(x) - \left(1 + \frac{m_i^*}{m}\right) \frac{d\Phi}{dx}}. \tag{8.5}$$

For $y = 0.2$ $\left(\text{i.e., } x = 0.2 \sqrt{\frac{m_i^*}{m_e}}\right)$, the errors thus introduced do not exceed 5% . If x is sufficiently large ($x \gtrsim 2$), then the denominator in expression (8.5) can be replaced by unity with sufficient accuracy. Then, equating this expression to unity, we find the value of the parameter y, at which the mean energy losses in the electrons equals the corresponding losses in the ions. This value is determined by the equation

$$\frac{4}{3\sqrt{\pi}} \left| \frac{m_i^* e_e}{m_e e_i^*} \right| y^3 = 1 + \frac{2}{\sqrt{\pi}} \left| \frac{m_i^* e_e}{m e_i^*} \right| y. \tag{8.6}$$

The equations of applicability for the latter equation are $y \ll 1$ and $x \gg 1$, or in view of the first relationship (8.3),:

$$\sqrt{\frac{m_e}{m_i^*}} \ll y \ll 1. \tag{8.7}$$

In terms of other variables,

$$\frac{m}{m_i^*} \ll \frac{\mathcal{E}}{T^*} \ll \frac{m}{m_e}. \tag{8.8}$$

In order to determine y, we shall extract the square root from both sides of equation (8.6). Since y is small, in the right part we can confine ourselves to the first powers of y. This gives

$$y = \frac{1}{\sqrt[3]{\frac{4}{3\sqrt{\pi}}\left|\frac{m_i^* e_e}{m_e e_i^*}\right| - \frac{2}{3\sqrt{\pi}}\left|\frac{m_i^* e_e}{m e_i^*}\right|}}. \tag{8.9}$$

The second term in the denominator is small as compared with the first. If it is discarded, then the condition (8.7) of applicability for formula (8.9) can be written in the form /113

$$\left(\frac{m_e}{m_i}\right)^{1/3} \ll \sqrt[3]{\frac{9\pi}{16}\left(\frac{e_i^*}{e_e}\right)^2} \ll \left(\frac{m_i^*}{m_e}\right)^{2/3} \tag{8.10}$$

and it can be assumed that it is always fulfilled.

Utilizing expression (8.3) for y, formula (8.9) can be rewritten in the form

$$\mathcal{E} = \frac{\frac{m}{m_e} T^*}{\sqrt[3]{\frac{4}{3\sqrt{\pi}}\left|\frac{m_i^* e_e}{m_e e_i^*}\right| - \frac{2}{3\sqrt{\pi}}\left|\frac{m_i^* e_e}{m e_i^*}\right|}}. \tag{8.11}$$

If the energy of a test ion is greater than this quantity, then the plasma electrons are heated to a greater extent; if it is smaller than this quantity, then the ions are heated to a greater extent. For a bundle of protons in a hydrogen plasma, we obtain $\mathcal{E} = 16T^*$ from formula (8.11), and for a bundle of deutrons in a deutron plasma - $\mathcal{E} = 20T^*$.

Table 4 lists the values of the relationship $\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_e : \left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{i^*}$ for a bundle of protons in a hydrogen plasma, and for a bundle of deutrons in a deutron plasma for different values of the parameter $x = \frac{\mathcal{E}}{T^*}$.

It can be seen from Table 4 that for a hydrogen plasma in the energy interval $\frac{3}{2} T^* < \mathcal{E} < 4T^*$ the energy transmitted to the ions exceeds the corresponding energy transmitted to the electrons by a /114

factor greater than ten. For a deuterium plasma, this applies to a wider interval $\frac{3}{2} T^* < \mathcal{E} < 5T^*$.

TABLE 4

VALUE OF THE RELATIONSHIP $\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_e : \left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{i^*}$ FOR DIFFERENT VALUES OF THE PARAMETER $\frac{\mathcal{E}}{T^*}$

$\frac{\mathcal{E}}{T^*}$	$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_e : \left\langle \frac{d\mathcal{E}}{dt} \right\rangle_{i^*}$	
	A Bundle of Protons in a Hydrogen Plasma	Bundle of Deutrons in a Deutron Plasma.
1,5	0	0
2	0,024	0,017
3	0,058	0,041
4	0,097	0,069
5	0,14	0,10
10	0,48	0,34
20	1,5	1,0
30	2,8	1,9
40	4,4	3,1
50	6,0	4,2
100	17	12
1 000	410	290
10 000	1800	3400

3. The considerations set forth in Section 2 are of great importance in understanding the principles underlying the operation of prospective *thermonuclear reactors*. For example, in a reactor which will operate with a *deuterium plasma*, the following *nuclear reactions* must take place:

$$d + d \rightarrow \mathrm{He}^3 + n + 3.25\ \mathit{Mev},$$

$$d + d \rightarrow t + p + 4\ \mathit{Mev}.$$

The charged products of these reactions p, t and He^3 have an initial energy of 3, 1 and 0.81 *Mev*, respectively. Interacting with the plasma, they transmit energy to it, heating its electrons and ions. Neutrons leave the plasma, and carry away energy with them. The *heating of the ions*, and not of electrons, is important for the operation of the reactor. However, in actuality the reverse occurs. At first, the charged products of thermonuclear reactions heat plasma

electrons primarily. Only after their energy is decreased within a definite limit, are primarily plasma ions heated.

For example, let us consider protons which have the greatest energy of all the products from nuclear reactions. Utilizing formula (8.11), we can readily calculate that the rates of energy transfer from protons to electrons and to deutrons in the plasma are identical for a proton energy $\mathcal{E}_p = 10.5T^*$ which comprises $\mathcal{E}_p \sim 525$ kev for a plasma temperature of $T^* = 50$ kev. With an initial energy of the protons $\mathcal{E}_p = 3$ Mev, the transfer of energy to the electrons takes place approximately 16 times more rapidly than to the plasma deutrons. The maximum transfer of energy from a bundle of protons to deutrons in the plasma, as can be seen from Table 3, takes place at a proton energy of $\mathcal{E}_p = 2.03T^*$ which comprises $\mathcal{E}_p \approx 100$ kev at $T^* = 50$ kev. As can be readily verified with the aid of formula (7.1), this maximum rate approximately equals

$$\left\langle \frac{d\mathcal{E}_p}{dt} \right\rangle_d = -\frac{2\pi L e^4 n}{\sqrt{2m_p \mathcal{E}_p}}.$$

For $\mathcal{E}_p = 100$ kev, $n = 10^{13}$ proton/cm^3, $L = 22$, and we thus obtain

$$\left\langle \frac{d\mathcal{E}_p}{dt} \right\rangle_d = -63 \text{ KEV/SEC}$$

4. In conclusion, let us briefly consider the case in which a bundle is composed of electrons having an energy $\mathcal{E}_e$. In this case, in formulas (8.1) and (8.2) $m = m_e$, $e = e_e$, and consequently /115

$$y = \sqrt{\frac{\mathcal{E}_e}{T^*}}, \quad x = y\sqrt{\frac{m_i^*}{m_e}}.$$

It is assumed that the energy of an electron $\mathcal{E}_e$ exceeds the critical energy when it interacts both with electrons and with ions in a plasma. As can be seen from Table 3, from the first assumption it follows that $\mathcal{E}_e > T^*$, i.e., $y > 1$. Consequently, the parameter x is large, and the expression in the square brackets in formula (8.1) equals unity to a great degree of accuracy. If $y \gg 1$ (which is practically sufficient for $y > 2$), then it is possible to replace the square brackets in formula (8.2) by unity. As a result, we obtain

$$\left\langle \frac{d\mathcal{E}_e}{dt} \right\rangle_e : \left\langle \frac{d\mathcal{E}_e}{dt} \right\rangle_{i^*} = \left| \frac{m_i^* e_e}{m_e e_i^*} \right|.$$

The energy of the electron bundle is transmitted to the electrons in the plasma $\left|\frac{m^*_i e}{m e^*_i}\right|$ *times more rapidly than to the ions, i.e., prac-*

tically all of the energy is transmitted to the electrons.

9. EQUALIZATION OF TEMPERATURES IN A TWO-COMPONENT PLASMA

1. In Sections 7 and 8, we discussed the exchange of energy between a monoenergetic bundle of similar charged particles, and one of plasma components, consisting of identical particles with Maxwell velocity distribution. Let us now study the case in which - instead of a monoenergetic bundle - a group of identical plasma particles is chosen, whose velocity at a certain moment of time has a Maxwell distribution with a temperature T. We shall study the exchange of energy between this group of particles and any other group of plasma particles consisting also of identical particles, whose velocity at the same moment of time is characterized by a Maxwell distribution with the temperature T^*. At the moment of time being considered, it is necessary to determine the mean energy Q which is transmitted per unit of time from one particle of the first group to all of the particles of the second group. This quantity can be found from expression (7.1), if the sign is changed in it, and if then one averages over the Maxwell distribution of the particle velocities in the first group. Introducing the notation

$$b = \sqrt{\frac{m}{2T}}, \tag{9.1}$$

we have

$$\left\langle \frac{\Phi(b^*v)}{v} \right\rangle = \frac{4b^3}{\sqrt{\pi}} \int_0^\infty \Phi(b^*v)\, e^{-b^2v^2} v\, dv =$$

$$= \frac{8b^3}{\pi b^{*2}} \int_0^\infty e^{-\left(\frac{b}{b^*}x\right)^2} x\, dx \int_0^x e^{-\xi^2} d\xi.$$

The integral can be readily calculated by interchanging the order of integration with respect to ξ and with respect to x: /116

$$\left\langle \frac{\Phi(b^*v)}{v} \right\rangle = \frac{8b^3}{\pi b^{*2}} \int_0^\infty e^{-\xi^2} d\xi \int_\xi^\infty e^{-\left(\frac{b}{b^*}x\right)^2} x\, dx = \frac{2}{\sqrt{\pi}} \frac{bb^*}{\sqrt{b^2+b^{*2}}}.$$

In addition,

$$\langle e^{-b^{*2}v^2} \rangle = \left(\frac{b}{\sqrt{\pi}}\right)^3 \int_0^\infty e^{-(b^2+b^{*2})v^2} 4\pi v^2\, dv = \left(\frac{b}{\sqrt{b^2+b^{*2}}}\right)^3.$$

Utilizing these results, we find the following after simple calculations:

$$Q = \frac{\frac{3}{2}(T-T^*)}{\tau_{\mathcal{E}}}, \tag{9.2}$$

where

$$\tau_{\mathcal{E}} = \frac{3mm^*}{8\sqrt{2\pi}n^*L(ee^*)^2}\left(\frac{T}{m}+\frac{T^*}{m^*}\right)^{3/2}. \qquad (9.3)$$

Formulas (9.2) and (9.3) were first obtained by L. Spittser (Ref. 8) by different methods, and by V. I. Kogan, independently of him (Ref. 9). For the partial case, when $m^* << m$, these formulas were previously introduced by L. D. Landau (Ref. 10, Ref. 11) [see formula (9.13)]. In Section 17, another derivation of formulas (9.2) and (9.3) will be given, based on the kinetic equation for a plasma.

The quantity $\tau_{\mathcal{E}}$ has the dimension of time. It can be called the *time for temperature equalization* of the two groups of particles being considered.

As must be the case, according to formula (9.2), the energy is always transmitted from the group of particles which is heated to the greatest extent to the group of particles which is heated to the least extent, and ceases when the temperatures of these groups are equalized. We can designate the temperature T, at which the rate of energy transfer is maximum, by T_m (it is assumed that the temperature T^* is kept constant). We can find the quantity T_m if the derivative $\frac{dQ}{dT}$ is set equal to zero at a fixed temperature T^*. In this way we obtain

$$T_m = \left(3+2\frac{m}{m^*}\right)T^*. \qquad (9.4)$$

2. Let us apply the results obtained to a *two-component quasi neutral plasma*, which consists of electrons and uniform, positively-charged ions. The question then arises of whether a *quasi-equilibrium state* is possible, and under what conditions; in this state the velocities both of the electrons and of the ions in the plasma have a Maxwell distribution, but with different temperatures T_e and T_i. In order to provide an accurate solution of this question, it will be necessary to examine the process by which thermodynamic equilibrium is established in a plasma, starting from an arbitrary, initial state. However, it is possible to make a simpler postulation. We can use $\tau_{\mathcal{E}}^{ee}$ to designate the *time required to establish* the Maxwell distribution of the electrons, which results from the internal interactions between the electrons themselves. We can designate the analogous quantity for ions by $\tau_{\mathcal{E}}^{ii}$. Finally, we can designate the *time required to establish* the thermodynamic equilibrium between electrons and ions by $\tau_{\mathcal{E}}^{ei}$. It is sufficient that the following two conditions

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be fulfilled in order that the desired quasi-equilibrium state come into being for a random initial state of the plasma:

$$\tau_{\mathcal{E}}^{ee} \ll \tau_{\mathcal{E}}^{ei}, \quad \tau_{\mathcal{E}}^{ii} \ll \tau_{\mathcal{E}}^{ei}. \tag{9.5}$$

In the opposite case for random initial states of the plasma, such a quasi-equilibrium state cannot come into being. It is only possible for special initial states. The times $\tau_{\mathcal{E}}^{ee}$, $\tau_{\mathcal{E}}^{ii}$, $\tau_{\mathcal{E}}^{ei}$, can be estimated with the aid of formula (9.3), and the given question can be solved qualitatively.

The time $\tau_{\mathcal{E}}^{ei}$ can be identified with the time of equalization for the electron and ion temperatures, which is given by expression (9.3). If the ions are singly charged, and the plasma is quasi neutral ($n = n^*$), then expression (9.3) is symmetrical with respect to ions and electrons. Therefore we can set, for example, $T = T_e$, $T^* = T_i$, $m = m_e$, $m^* = m_i$. In addition, expression (9.3) can be simplified, by utilizing the smallness of the relationship $\frac{m_e}{m_i}$. If the condition is fulfilled $\frac{T_i}{m_i} \ll \frac{T_e}{m_e}$, i.e.

$$T_i \ll \frac{m_i}{m_e} T_e, \tag{9.6}$$

then we obtain from formula (9.3)

$$\tau_{\mathcal{E}}^{ei} \approx \frac{3 m_i T_e^{3/2}}{8 \sqrt{2\pi m_e} n L e^4}. \tag{9.7}$$

The times $\tau_{\mathcal{E}}^{ee}$ and $\tau_{\mathcal{E}}^{ii}$ can be estimated in the following way. Let us divide all of the electrons in the plasma into two groups with the concentrations $\frac{n}{2}$ each. Let us assume that the velocities of the electrons in each of the groups has a Maxwell distribution, but with different temperatures T_e and T_e'. Then formulas (9.2) and (9.3) can be used to describe the energy exchange of these two groups. The equalization time of their temperatures can be identified with the time $\tau_{\mathcal{E}}^{ee}$ required to establish the Maxwell distribution. Let us introduce the unimportant assumption that $T_e \gg T_e'$. Then, substituting the quantities $n^* = \frac{n}{2}$, $m = m^* = m_e$, $T = T_e$ in formula (9.3) and disregarding $T^* \equiv T_e'$ as compared with T_e, we obtain /118

$$\tau_{\mathcal{E}}^{ee} \approx \frac{3 \sqrt{m_e} T_e^{3/2}}{4 \sqrt{2\pi}\, n L e^4}. \tag{9.8}$$

Although this formula is obtained for an absolutely definite, specialized initial velocity distribution of the electrons, there is no doubt that it is suitable for a qualitative approximation of the time required to establish Maxwell distribution for any (not too specialized) initial velocity distribution of the electrons. Thus, T_e can designate the temperature of the electrons, for example, in the final state, when the Maxwell distribution will have been achieved.

In this way we can find the following for ions

$$\tau_{\mathcal{E}}^{ii} \approx \frac{3\sqrt{m_i}T_i^{3/2}}{4\sqrt{2\pi}\, nLe^4}. \tag{9.9}$$

In order to determine the order of magnitude of $\tau_{\mathcal{E}}^{ee}$, $\tau_{\mathcal{E}}^{ii}$, $\tau_{\mathcal{E}}^{ei}$, let us obtain their values for a deuterium plasma for $T_e = T_i = 10$ kev $= 1.6 \cdot 10^{-8}$ erg and $n = 10^{14}$particle/cm^3. Utilizing formulas (9.8), (9.9) and (9.7), and also Tables 1 and 2, we obtain

$$\tau_{\mathcal{E}}^{ee} \approx 2 \cdot 10^{-4} \text{ sec}; \quad \tau_{\mathcal{E}}^{ii} \approx 1 \cdot 10^{-2} \text{ sec}; \quad \tau_{\mathcal{E}}^{ei} \approx 0{,}36 \text{ sec}.$$

Thus, in the case being considered the conditions (9.5) are fulfilled. In the general case, in order of magnitude

$$\tau_{\mathcal{E}}^{ee} : \tau_{\mathcal{E}}^{ii} : \tau_{\mathcal{E}}^{ei} \approx 1 : \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{T_i}{T_e}\right)^{3/2} : \frac{m_i}{m_e}. \tag{9.10}$$

It is thus assumed that condition (9.6) is fulfilled. We can see that under this condition $\tau_{\mathcal{E}}^{ee} \ll \tau_{\mathcal{E}}^{ei}$ always holds. If, in addition, condition $\left(\frac{T_i}{T_e}\right)^{3/2} \ll \left(\frac{m_i}{m_e}\right)^{1/2}$, i.e.,

$$T_i \ll \left(\frac{m_i}{m_e}\right)^{1/3} T_e, \tag{9.11}$$

then $\tau_{\mathcal{E}}^{ii} \ll \tau_{\mathcal{E}}^{ei}$ will also always hold.

Inequality (9.6) follows from inequality (9.11). Therefore, if condition (9.11) is fulfilled, condition (9.5) is also fulfilled. In addition, it will be shown in Section 12 that during a period of time, whose order of magnitude is determined by formulas (9.8) and (9.9), the anisotropic velocity distributions of the ions and electrons can change into isotropic distributions. It thus follows that *condition (9.11) is sufficient for assuming that the velocity distribution of the ions and electrons is almost a Maxwell distribution thoughout the entire process of energy exchange between ions and electrons - with the exception, possibly, of its brief initial stage. In particular, this will occur for* $T_i < T_e$.

Thus, *the process by which equilibrium is established between the ion and electron components of a plasma is much slower than the process by which the Maxwell distribution is established for only the electrons or for only the ions. Therefore, in a plasma with a random initial velocity distribution of electrons and ions, the Coulomb collisions very rapidly lead to the establishment of almost Maxwell velocity distributions of electrons and ions, while the establishment of thermal equilibrium between the electrons and the ions takes place much later. As a result, a quasi-equilibrium state arises, which is characterized by two temperatures: electron temperature* T_e *and ion temperature* T_i *. Condition (9.11) was explicitly used in the derivation. If this condition is not fulfilled, then, generally speaking, this conclusion is not valid.* /119

However, it would be incorrect to state that condition (9.11) is not only sufficient, but also necessary for the formation of such a quasi-equilibrium state of the plasma. Under certain initial conditions, it can even arise when condition (9.11) is not fulfilled. For example, let us assume that the initial distribution of the ions is a Maxwell distribution and that condition (9.6) is fulfilled, from which it follows $\tau_{\mathcal{E}}^{ee} \ll \tau_{\mathcal{E}}^{ei}$. Then Maxwell velocity distribution of the electrons arises very rapidly, and there is no basis for expecting that the Maxwell distributions of ions and electrons will be greatly distorted as a result of the interactions between them. As a result, throughout the entire process of energy exchange between the ions and electrons, the velocity distribution of ions in electrons will be almost a Maxwell distribution, although condition (9.11) cannot be fulfilled.

3. The process by which the electron and ion temperatures are equalized can be described by formula (9.2), in which we must set $T = T_e$, $T^* = T_i$. In the case under consideration, in view of condition (9.6) it is possible to disregard the second component in the parenthesis in formula (9.3). Then we have

$$Q \equiv Q_{ei} = \frac{\frac{3}{2}(T_e - T_i)}{\tau_{\mathcal{E}}^{ei}}, \tag{9.12}$$

where Q_{ei} is the mean energy transmitted each second to one electron-ion component of the plasma, and, as was done previously, the time $\tau_{\mathcal{E}}^{ei}$ is determined by expression (9.7). As a result, we arrive at the formula

$$Q_{ei} = \frac{4\sqrt{2\pi m_e} nLe^4}{m_i T_e^{3/2}}(T_e - T_i), \tag{9.13}$$

which was first obtained by L. D. Landau by another method (Ref. 10,

Ref. 11). It can be represented in the following form:

$$Q_{ei} = \frac{A}{\sqrt{T_i}} \frac{\alpha - 1}{\alpha^{3/2}}, \tag{9.14}$$

where A does not depend on T_e and T_i, and the relationship /120
$\alpha = \frac{T_e}{T_i}$ is designated by α. The graph of this function is shown in Figure 5, for a fixed temperature T_i. It reaches a maximum for $\alpha = 3$, for $\alpha = 5$ it has a point of inflection, and then it slowly decreases with an increase in α. For large α, the quantity Q_{ei} changes inversely proportionally to the square root of α. For $\alpha \approx 25$, the quantity Q_{ei} equals half of its maximum value. For $\alpha = 2$, it comprises 0.92 of the maximum value. With a further decrease in α, the quantity Q_{ei} rapidly decreases, changing its sign for $\alpha = 1$.

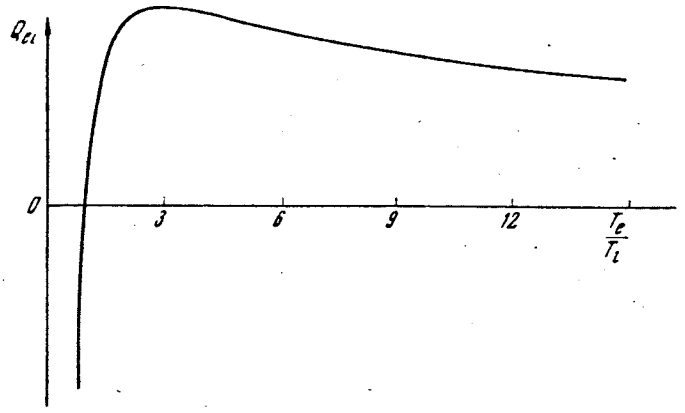

Figure 5.

4. In order to describe the process by which the electron and ion temperatures are equalized, in addition to equation (9.12), one more equation is necessary. It can be obtained from the *law of conservation of energy*. In the simplest case, when a plasma does not obtain and does not lose energy, this law can be reduced to $T_e + T_i = \text{const}$. Let us designate the final plasma temperature by T_∞, i.e., the temperature which is established in it when the temperature equalization is almost completed. Then

$$T_e + T_i = 2T_\infty. \tag{9.15}$$

Eliminating T_i from formula (9.12) with the aid of this relationship, and taking the fact into consideration that $Q_{ei} = -\frac{3}{2}\frac{dT_e}{dt}$, we obtain

$$\frac{T_e^{3/2}\,dT_e}{T_e - T_\infty} = -T_\infty^{3/2}\frac{dt}{\tau_\infty},$$

where

$$\tau_\infty = \frac{3m_i T_\infty^{3/2}}{16\sqrt{2\pi m_e} n L e^4}. \qquad (9.16)$$

After integration, we have

$$\ln\left|\frac{\sqrt{T_e}-\sqrt{T_\infty}}{\sqrt{T_e}+\sqrt{T_\infty}}\right| = -\frac{t}{\tau_\infty} - \frac{2}{3}\left(\frac{T_e}{T_\infty}\right)^{3/2} - 2\left(\frac{T_e}{T_\infty}\right)^{1/2} + C, \qquad (9.17)$$

where C is the constant determined from the initial conditions.

The curve of (9.17) is shown in Figure 6, for the case when $T_e > T_i$. The corresponding curve for ion temperatures is also shown. The counting of time begins at the moment when the temperature of the ions equals zero. The curves are suitable for determining T_e and T_i for any initial conditions. For this purpose, it should only be noted that on the time axis the point, at which T_e and T_i equal their initial values in the scale being used, is selected as the new beginning of time counting. The subsequent behavior of the temperatures T_e and T_i is described in Figure 6 by the branching of the curves located to the right of this point. /121

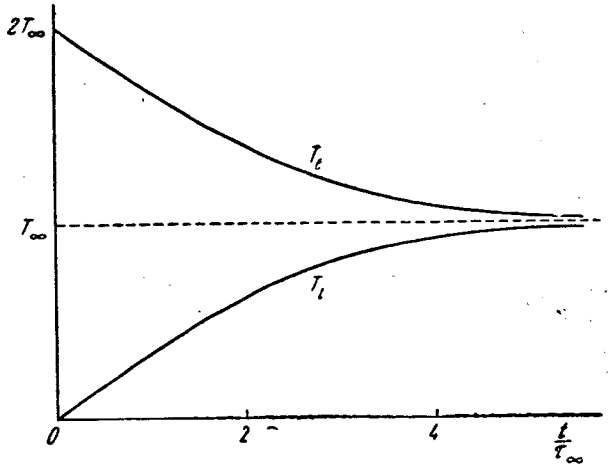

Figure 6.

Figure 7 shows the curves for the temperature change of electrons and ions for the case $T_e < T_i$. The sections of the curves where condition (9.11) is not fulfilled are shown by dashed lines. The plasma states, corresponding to these sections of the curves, are not completely devoid of any meaning, because theoretically a quasi-equilibrium

state is possible with Maxwell velocity distributions of electrons and ions even when condition (9.11) is not fulfilled.

On the other hand, the initial sections of the curves in Figures 6 and 7 should be regarded as extrapolations, which do not correspond to reality, because the theory is not applicable if there is a very big difference between T_e and T_i.

For very large values of t, when the temperatures T_e and T_i differ very little from each other, formula (9.17) changes into

$$|T_e - T_\infty| = |T_i - T_\infty| = \text{const} \cdot e^{-\frac{t}{\tau_\infty}}. \qquad (9.18)$$

In this region, the temperature difference between T_e and T_i decreases by e times during the time τ_∞. The simple physical meaning of the time τ_∞ is established in this way. /122

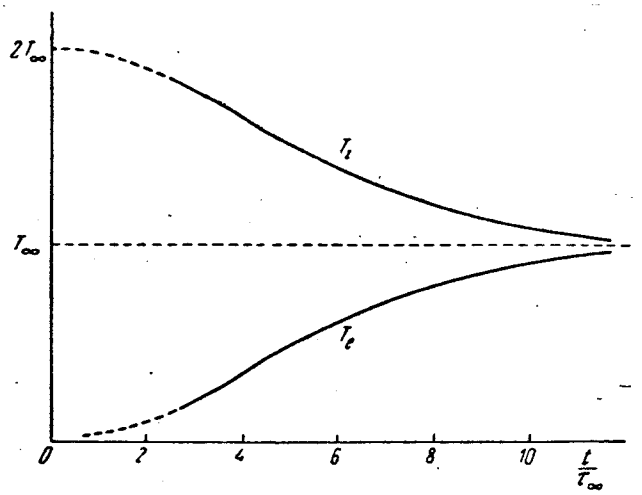

Figure 7.

10. IMPULSE CHANGE OF A TEST PARTICLE MOVING IN A PLASMA

1. The formulas for a change in the kinetic energy of a test particle moving in a plasma were derived in Section 6. Under the same assumptions, we shall now study the *impulse change* of a test particle. We take as our point of departure formula (3.12). For an isotropic velocity distribution of field particles, the vector $\mathbf{E}_v$ is parallel to the vector $\mathbf{v}$, and is directed in the same sense. Therefore, formula (3.12) determines the *longitudinal retardation* of a test particle, on the average. In addition to the longitudinal retardation, the *lateral deviations* of a particle from the direction

of its motion are also of interest. Naturally, they have an irregular nature. Therefore, we shall supplement formula (3.12) by an expression which determines the mean rate of change of the square of a transverse impulse. This expression can be found from formula (3.12) by utilizing formula (3.11).

Let us determine the change in the square of the impulse p for Coulomb collisions. During the collisions, both the longitudinal and the transverse components of the vector p change (with respect to the direction of motion of the test particle before the collision). In accordance with this:

$$\frac{dp^2}{dt} = \frac{dp_\parallel^2}{dt} + \frac{dp_\perp^2}{dt}. \tag{10.1}$$

Since $p^2 = 2m\mathcal{E}$, then

/123

$$\frac{dp^2}{dt} = 2m\frac{d\mathcal{E}}{dt}. \tag{10.2}$$

In addition,

$$\frac{dp_\parallel^2}{dt} = 2p_\parallel \frac{dp_\parallel}{dt} = 2\mathbf{p}\frac{d\mathbf{p}}{dt}. \tag{10.3}$$

Substituting these expressions in the preceding relationship, and averaging with respect to a group of test particles with a fixed impulse p, we obtain

$$\left\langle \frac{dp_\perp^2}{dt} \right\rangle = 2m\left\langle \frac{d\mathcal{E}}{dt} \right\rangle - 2\mathbf{p}\left\langle \frac{d\mathbf{p}}{dt} \right\rangle. \tag{10.4}$$

Utilizing formulas (3.11) and (3.12) , we find

$$\left\langle \frac{dp_\perp^2}{dt} \right\rangle = 2\Sigma^* L\varphi_v. \tag{10.5}$$

This formula describes the statistical process of the lateral deviation of a test particle.

2. The following interesting corollary can be most readily understood with the aid of an electrostatic analogy. This corollary was first noted, apparently, by S. T. Belyayev and G. I. Budker (Ref. 12). If the distribution function $f^*(v^*)$ is isotropic, i.e., it depends only on the magnitude of the velocity v^* , but not on its direction, then the "analog of density of electric charges in velocity space" - which is determined by expression (3.15) - has the same property. These "charges" are thus distributed in velocity space spherically

symmetrically around the origin $\mathbf{v}^* = 0$. As is known from electrostatics, in this case "the charges", which are further away from the origin than v, have no effect on the "electric field" $\mathbf{E}_v$ at the point $\mathbf{v}^* = \mathbf{v}$. Therefore, not changing the values of the integral (3.13), we can take the space within the sphere $v^* < v$ as the region of integration, which was done in the first formula (6.1). It thus follows that *the field particles, whose velocity exceeds the velocity of a test particle, have no effect on the value of* $\left\langle \frac{dp}{dt} \right\rangle$, *i.e. on the mean longitudinal retardation of this particle.*

This result is a corollary of the approximation being used, in which the Coulomb logarithm L does not depend on the relative velocity $\mathbf{u}$. But even in this approximation, field particles with velocities of $v^* > v$ have an effect on the lateral deviations of a test particle, as can be seen from formula (10.5) and (3.14): the integral

$$\int\limits_{v^*>v} \frac{1}{u} \varrho_v(\mathbf{v}^*)\, d\mathbf{v}^* = \int\limits_{v^*>v} \frac{\varrho_v(\mathbf{v}^*)}{v^*} d\mathbf{v}^*$$

differs from zero and is essentially positive. These particles have an effect on the kinetic energy of a test particle, always increasing it on the average, as follows from formula (3.11), and also from simple physical considerations. /124

3. Let us now calculate the Maxwell distribution (6.2) of field particles. In this case, $\mathbf{E}_v$ and φ_v are determined by the expressions (6.3) and (6.4). Substituting them in formulas (3.12) and (10.5), we find

$$\left\langle \frac{d\mathbf{p}}{dt} \right\rangle = -\frac{4\pi e^2}{v^3} \mathbf{v} \sum\nolimits^* \frac{Le^{*2}}{\mu} n^* \Phi_1(b^* v), \tag{10.6}$$

$$\left\langle \frac{dp_\perp^2}{dt} \right\rangle = \frac{8\pi e^2}{v} \sum\nolimits^* Ln^* e^{*2} \Phi(b^* v). \tag{10.7}$$

11. THE RANGE OF A RAPID ION IN A PLASMA

1. Let us assume that a rapid ion moves in a plasma with a kinetic energy which is much greater than the mean kinetic energy of thermal motion of ions and electrons in a plasma. In this case, $b^* v \gg 1$; therefore, it can be assumed that $\Phi_1(b^* v) = \Phi(b^* v) = 1$. Let us first disregard the change in direction of motion of a rapid (test) ion, which is described by formula (10.7), and we shall assume that

it moves in a strictly rectilinear manner. Thus, the ion undergoes retardation which is described by formula (10.6). If p and v are used to designate the mean (with respect to a group of test ions) values of its impulse and velocity, then this formula can be represented in the form

$$m\frac{dv}{dt} = -\frac{A}{v^2}, \tag{11.1}$$

where A is a constant which equals

$$A = 4\pi e^2 \sum^* \frac{Le^{*2}}{\mu} n^*. \tag{11.2}$$

Integrating equation (11.1), we obtain

$$v_0^3 - v^3 = \frac{3A}{m} t, \tag{11.3}$$

where v_0 is the value of the velocity v for $t = 0$.

If we introduce the path $dx = vdt$, which is traversed by a test ion during the time dt, then equation (11.1) can be written in the form

$$m\frac{dv}{dx} = -\frac{A}{v^3}$$

and after integration, we have

$$x = \frac{m}{4A}(v_0^4 - v^4). \tag{11.4}$$

Formulas (11.3) and (11.4) can be used, since the velocity v is such that the kinetic energy of a rapid ion exceeds the mean kinetic energy of thermal motion for plasma electrons and ions. If /125
the initial velocity v_0 is very large, and the final velocity v approximates the mean velocity of thermal motion for plasma field ions, then the components v^3 and v^4 can be disregarded in formulas (11.3) and (11.4). In particular, formula (11.4) determines the mean range l of a rapid ion in a plasma, i.e., the mean distance traversed by it until the time when it reaches thermal equilibrium with the surrounding plasma. Thus, for the range l we obtain

$$l = \frac{mv_0^4}{4A} = \frac{mv_0^4}{16\pi e^2 \sum^* \frac{Le^{*2}}{\mu} n^*}. \tag{11.5}$$

Since the reduced mass for ion-ion collisions is very large as compared with the reduced mass for ion-electron collisions, in a calculation of l the influence of the ions can be disregarded, as compared with the influence of the electrons. Then formula (11.5)

is simplified and assumes the following form:

$$l = \frac{mm_e}{16\pi e^2 e_e^2 Ln_e^*} v_0^4, \tag{11.6}$$

where the notation is the same as that used in Section 8.

Thus, *in Coulomb interactions, the retardation of rapid ions takes place primarily due to their scattering by plasma electrons.* This result, which appears to be paradoxical at first glance, agrees with the results obtained in Sections 6 - 8, where the energy exchange between a rapid ion and a plasma was discussed. All of this can be explained by the characteristics of Coulomb interaction. As formulas (2.14) and (2.15) show, at one and the same scattering angle ϑ, the impulse and energy increases δp, $\delta \mathcal{E}$ of a test particle are proportional to the first power of the reduced mass μ, as a result of the single scattering process. However, according to the Rutherford formula (3.5), the differential scattering cross-section $\sigma(\vartheta, u)$ depends on μ to a greater extent; it is inversely proportional to the square of μ. As follows from formula (6.7), the mean energy loss of a rapid particle per unit of time decreases with an increase in its velocity v: it is inversely proportional to v.

2. The assumption used in deriving formula (11.5) still remains to be verified. This assumption states that the trajectory of a test ion, while its velocity exceeds the mean velocity of thermal motion of plasma ions, is practically rectilinear. To prove this, we shall represent equation (10.7) in the form

$$\frac{dp_\perp^2}{dt} = \frac{B}{v},$$

where $B = 8\pi e^2 \Sigma^* Ln^* e^{*2}$ is the new constant equaling $m_e A$, correct to the coefficient on the order of two. Introducing the velocity v as an independent variable, and utilizing equation (11.1), we obtain /126

$$\frac{dp_\perp^2}{dt} = \frac{dp_\perp^2}{dv}\frac{dv}{dt} = -\frac{A}{mv^3}\frac{dp_\perp^2}{dt}.$$

Consequently,

$$\frac{dp_\perp^2}{dv} = -\frac{mB}{A}v.$$

After integration, we have

$$p_\perp^2 = \frac{Bm}{2A}(v_0^2 - v^2) < \frac{Bm}{2A}v_0^2.$$

Thus, if p_0 is the value of the initial impulse of a test ion, then

$$\left\langle \frac{p_\perp^2}{p_0^2} \right\rangle < \frac{B}{2Am} \approx \frac{m_e}{m}.$$

For the ions, $m >> m_e$ and therefore $\langle p_\perp^2 \rangle << p_0^2$. This means that the lateral shifts of the rapid ions are negligible; before they are slowed down, such ions move almost in a rectilinear manner. This conclusion is not valid for electrons.

12. RELAXATION TIMES AND MEAN FREE PATHS

1. On the basis of the results derived in Section 10, let us introduce the concept of different *relaxation times*, with the aid of which qualitative statements can be readily made with respect to the extent to which a plasma approximates the state of thermodynamic equilibrium. As was done previously, let us examine an individual test particle in a plasma which has an arbitrary initial velocity v. We shall assume that the plasma itself is in a state of thermodynamic equilibrium. As the test particle moves, it is always retarded, on the average, and undergoes lateral deviations. Therefore, two types of relaxation time can be introduced: *the time of longitudinal retardation* $\tau_\parallel$ *and the time of lateral deviation* $\tau_\perp$. The first - $\tau_\parallel$ - determines the mean time in order of magnitude, during which a test particle loses ordered velocity in the direction of its initial motion. The second - $\tau_\perp$ - determines the mean time in order of magnitude, during which the velocity vector of a test particle turns at an angle on the order of 90°. If p is the initial impulse of the particle, then it is reasonable to determine these times with the aid of the relationships

$$\left\langle \frac{dp_\parallel}{dt} \right\rangle = -\frac{p}{\tau_\parallel}, \tag{12.1}$$

$$\left\langle \frac{dp_\perp^2}{dt} \right\rangle = \frac{p^2}{\tau_\perp}. \tag{12.2}$$

Comparing them with formulas (10.6) and (10.7), we find /127

$$\tau_\parallel = \frac{mv^3}{4\pi e^4 \sum^* \frac{Le^{*2}}{\mu} n^* \Phi_1 (b^* v)}; \tag{12.3}$$

$$\tau_\perp = \frac{m^2 v^3}{8\pi e^4 \Sigma^* Le^{*2} n^* \Phi (b^* v)}. \tag{12.4}$$

2. Let us mentally select some plasma component - for example,

one consisting of electrons - and let us disregard the interaction of the test particle with all the other components. Then it is possible to introduce the relaxation time τ^* in order to describe the interaction of the test particle with the separate plasma component. For example,

$$\tau_{\parallel}^{*} = \frac{m\mu v^3}{4\pi (ee^*)^2 Ln^* \Phi_1 (b^* v)}.$$

The total relaxation time τ, which describes the interaction of the test particle with the entire plasma, can be represented in the following form - as the structure of formulas (12.3) and (12.4) show:

$$\frac{1}{\tau} = \sum^{*} \frac{1}{\tau^*}, \tag{12.5}$$

where the summation is done for all types of field particles.

For purposes of simplicity, let us assume that the plasma is a two-component plasma, and consists of electrons and positive, singly-charged ions. Let us assume that an electron is the test particle. Then we can speak of four relaxation times, which we shall designate by $\tau_{\parallel}^{ee}$ $\tau_{\perp}^{ee}$ $\tau_{\parallel}^{ei}$ $\tau_{\perp}^{ei}$. The first two of these refers to the interaction of a test electron with an electron plasma component, and the last two - to the interaction of the test electron with the ion component. In addition, let us assume that the velocity of the test electron is greater than, or on the order of, the mean velocity of thermal motion for plasma electrons. Also we shall assume that the thermal velocities of the ions - as is almost always the case - are less than the thermal velocities of the electrons. Then we can assume that in formulas (12.3) and (12.4) the functions Φ_1 and Φ equal unity, and we obtain

$$\tau_{\parallel}^{ee} = \tau_{\perp}^{ee} = \frac{1}{2}\tau_{\parallel}^{ei} = \tau_{\perp}^{ei} = \frac{m_e^2 v_e^3}{8\pi e^4 Ln}, \tag{12.6}$$

where the concentration of electrons or the concentration of positive plasma ions (quasi-neutral), which is equal to it, is designated by n. The relaxation times which are introduced describe the behavior of any definite *individual* test particle which has a definite velocity v. Let us now introduce the relaxation times which characterize *the mean properties of the plasma as a whole.* They can be obtained from the relaxation times which have already been introduced, for individual particles, by averaging with respect to the corresponding distributions of their velocities. Therefore, we shall designate the new relaxation times by the same letters, but with short lines above - for example, /128
$\bar{\tau}_{\parallel}^{ee}$, $\bar{\tau}_{\perp}^{ee}$, $\bar{\tau}_{\parallel}^{ei}$ etc. If we are interested in the behavior of the electron plasma component, then the corresponding relaxation times can be

obtained from the expression (12.6), substituting any mean thermal velocity of the electron, instead of the velocity v_e. The former can be determined, for example, by the relationship $m_e v_e^2 = 3T_e$. Then the formulas (12.6) are transformed into

$$\bar{\tau}_{\parallel}^{ee} = \bar{\tau}_{\perp}^{ee} = \frac{1}{2}\bar{\tau}_{\parallel}^{ei} = \bar{\tau}_{\perp}^{ei} = \frac{3\sqrt{3m_e}T_e^{3/2}}{8\pi nLe^4}. \tag{12.7}$$

These quantities differ from the equalization time $\tau_{\mathcal{E}}^{ee}$ of the temperatures of electron plasma components (9.8) by the numerical multiplier $\sqrt{\frac{3}{2\pi}} \approx 0.7$. Thus, *all of the times* $\bar{\tau}_{\parallel}^{ee}$ $\bar{\tau}_{\perp}^{ee}$ $\bar{\tau}_{\parallel}^{ei}$ $\bar{\tau}_{\perp}^{ei}$ $\tau_{\mathcal{E}}^{ee}$ *agree with each other in order of magnitude*; therefore, they can be combined into one - *the mean electron relaxation time*

$$\bar{\tau}_e = \frac{3\sqrt{3m_e}}{8\pi nLe^4}T_e^{3/2}, \tag{12.8}$$

which characterizes the behavior of the plasma on the whole.

In a similar manner, we can obtain the following for ions in order of magnitude:

$$\bar{\tau}_{\parallel}^{ii} = \bar{\tau}_{\perp}^{ii} = \frac{3\sqrt{3m_i}}{8\pi nLe^4}T_i^{3/2}. \tag{12.9}$$

These times differ from expression (9.9) for the time $\tau_{\mathcal{E}}^{ii}$ of equalization of the temperatures of ion plasma components also by the numerical multiplier $\sqrt{\frac{3}{2\pi}}$. Thus, *all the times* $\bar{\tau}_{\parallel}^{ii}$ $\bar{\tau}_{\perp}^{ii}$ and $\tau_{\mathcal{E}}^{ii}$ *agree with each other in order of magnitude*, and can be combined into one - *the mean ion relaxation time*

$$\bar{\tau}_i = \frac{3\sqrt{3m_i}}{8\pi nLe^4}T_i^{3/2}, \tag{12.10}$$

which also characterizes the conduct of the plasma as a whole.

The mean relaxation times $\bar{\tau}_{\parallel}^{ie}$ and $\bar{\tau}_{\perp}^{ie}$, remain to be examined. They refer to the processes of interaction between ions and electrons in a plasma. We shall regard a plasma ion, which moves with a mean thermal velocity v, as a test ion. If the velocity v - as is almost always the case - is small as compared with the thermal electron velocity, then the argument $x = b^*v$ is small, and the functions $\Phi(x)$ and $\Phi_1(x)$ can be replaced by the first terms of their expansion in power series.

$$\Phi(x) = \frac{2}{\sqrt{\pi}} x,$$

$$\Phi_1(x) = \frac{4}{3\sqrt{\pi}} x^3.$$

We then find from formulas (12.3) and (12.4): /129

$$\bar{\tau}_{\parallel}^{ie} = \frac{3m_i T_e^{3/2}}{8\sqrt{2\pi m_e}\, nLe^4}, \tag{12.11}$$

$$\bar{\tau}_{\perp}^{ie} = \frac{3m_i T_i T_e^{1/2}}{8\sqrt{2\pi m_e}\, nLe^4}. \tag{12.12}$$

The time $\bar{\tau}_{\parallel}^{ie}$ agrees with the time τ_{g}^{ei} of equalization of the temperatures of electron and ion plasma components, as can be seen from a comparison of expressions (12.12) and (9.7). With respect to the time $\bar{\tau}_{\perp}^{ie}$, in order of magnitude it agrees with $\bar{\tau}_{\parallel}^{ie}$, if only the temperatures T_i and T_e do not differ from each other too much. On the other hand, the time $\bar{\tau}_{\perp}^{ie}$ does not play a significant role, since - in the majority of cases - it is $\bar{\tau}_{\perp}^{ii}$ and not it, which is a decisive factor in the examination of the process of lateral ion deviation. Actually, from formulas (12.9) and (12.12), we obtain

$$\frac{\bar{\tau}_{\perp}^{ii}}{\bar{\tau}_{\perp}^{ie}} \approx \sqrt{\frac{m_e}{m_i}\frac{T_i}{T_e}}. \tag{12.13}$$

If $T_i < \frac{m_i}{m_e} T_e$, then $\bar{\tau}_{\perp}^{ii} < \bar{\tau}_{\perp}^{ie}$. The lateral deviation which is caused by collisions with ions is a more rapid process than the deviation caused by collisions with electrons. Our statements substantiate this. Therefore, it is sufficient to use one time $\bar{\tau}_{ie}$, which can be called *the mean ion-electron relaxation time* and can be determined by the expression

$$\bar{\tau}_{ie} = \frac{3m_i T_e^{3/2}}{8\sqrt{2\pi m_e}\, nLe^4}. \tag{12.14}$$

Comparing expressions (12.8), (12.10) and (12.14), we obtain

$$\bar{\tau}_e : \bar{\tau}_i : \bar{\tau}_{ie} \approx 1 : \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{T_i}{T_e}\right)^{3/2} : \frac{m_i}{m_e}, \tag{12.15}$$

which agrees with the relationships (9.10).

3. All of the expressions for relaxation time, which have been obtained in the present section, refer to the Maxwell velocity distribution of electrons and ions. However, there is no reason to doubt the fact that they are suitable for qualitative approximations for non-Maxwell distributions.

These expressions make it possible to refine the statements regarding the extent to which the plasma approximates the state of thermodynamic equilibrium, which were discussed in Section 9. If the plasma has an arbitrary initial velocity distribution of electrons and ions, two processes begin as a result of the collisions between electrons and between electrons and ions. These processes take place at almost the same rates; one is the process by which isotropic velocity distribution of electrons is established, and the other is the process of energy exchange between them. Both of these processes very rapidly lead to an almost Maxwell velocity distribution of the electrons, with the temperature T_e which slowly changes with time. /130

If the mean energy of electrons and ions is such that the following condition is fulfilled

$$T_i \ll \left(\frac{m_i}{m_e}\right)^{1/3} T_e, \tag{12.16}$$

then $\bar{\tau}_{ii} \ll \bar{\tau}_{ie}$. If this condition is fulfilled, after the Maxwell distribution of electron velocities is established, two slower processes begin - which take place at almost the same velocities. These are the processes by which isotropic velocity distribution and Maxwell velocity distribution are established for the ions. As the result, a quasi-equilibrium state of the plasma arises, with Maxwell velocity distributions of electrons and ions, but with different electron T_e and ion T_i temperatures. Then, the slow process of equalization of electron and ion temperatures begins, which ultimately leads to equilibrium Maxwell distribution for the entire plasma. This process was discussed in detail in Section 9.

4. Along with the relaxation times, a useful concept in examining the many processes in a plasma is the concept of the *mean free path*. When applied to a plasma, this concept is just as clear as in the classical *kinetic theory of gases*. The comparative clearness of the concept of the mean free path of molecules in gases is due to the fact that the trajectory of a gas molecule takes the form of a broken line with sharp breaks which arise when the molecule under consideration collides with other gas molecules. Due to the slowness with which the Coulomb forces increase with distance, similar breaks which are caused by close interactions are encountered very rarely. The far-removed interactions, and not the close ones, play a basic role in the

direction change of particle motion in a plasma. Due to this fact, the trajectory of a particle in a plasma has the form, not of a broken line, but of a twisting line with constantly-decreasing curvature. Therefore, only conditional statements can be made about the mean free path in a plasma. This concept can take on an additional meaning, depending on which side of the particle motion we turn our attention. One of the possible definitions is as follows: *the mean free path* λ *in a plasma is the mean distance traversed when the direction of motion of the particle under consideration is changed by an angle on the order of* $90°$. In accordance with this, the quantity λ can be quantitatively determined with the aid of the relationship

$$\left\langle \frac{dp_\perp^2}{dx} \right\rangle = \frac{p^2}{\lambda}, \tag{12.17}$$

where dx is the element of the path length traversed by the test particle. Since $dx = vdt$, this equation can be rewritten in the form /131

$$\left\langle \frac{dp_\perp^2}{dt} \right\rangle = v\frac{p^2}{\lambda}.$$

Comparing this relationship with formula (12.2), we obtain

$$\lambda = v\tau_\perp. \tag{12.18}$$

Therefore, in view of formula (12.4), we have

$$\lambda = \frac{m^2v^4}{8\pi e^2 \Sigma^* l.e^{*2} n^* \Phi(b^* v)}. \tag{12.19}$$

It should also be possible to determine the mean free path by the relationship $\lambda' = v\tau_\parallel$. In every case, the mean free path refers to the behavior of any *individual* test particle which has the definite velocity v. For the characteristics of the plasma as a whole, it is possible to introduce the mean free paths λ or λ', which characterize the behavior of particles *of the plasma as a whole*. These quantities are obtained from the expressions for λ and λ' by replacing the individual velocity v by the corresponding mean velocity of thermal motion.

13. THE PHENOMENON OF ELECTRON ESCAPE

1. In the derivation of formula (10.6), it was assumed that there is a Maxwell velocity distribution in a plasma, and that external force fields have an effect upon it. Let us now assume that at a certain moment of time, when the velocity distribution is a Maxwell distribution, a homogeneous electric field $\mathbf{E}$ is switched on. Then, at least at first when the velocity distribution has not changed significantly, an

increase of the impulse of the test particle with time, which is caused by its collision with plasma particles, will be determined by the right part of the equation (10.6), as was done previously. However, eE must now be added to this part, with which the electric field E acts upon the test particle. As a result, we obtain

$$\left\langle \frac{dp}{dt} \right\rangle = -\frac{4\pi e^2}{v^3} v \sum^{*} \frac{Le^{*2}}{\mu} n^{*} \Phi_1 (b^{*} v) + eE. \qquad (13.1)$$

If the velocity v of a test particle is not too small, the first component in the right part increases - with respect to absolute magnitude - with an increase in the velocity v (the function $\frac{\Phi_1(x)}{x^2}$ for $x = 0$ becomes zero. With an increase in x, it increases, reaching a maximum for $x = 0.968$; then it begins to decrease, and assymptotically strives to zero for $x \to \infty$). Therefore, for rather large v, the particle will not be retarded on the average, but will be accelerated. The acceleration of heavy ions is insignificant, but the acceleration of light electrons can appear in the phenomenon of *electron escape*, which was first pointed out by Giovanelly (Ref. 13). /132

For a Maxwell velocity distribution, electrons will always be found in a plasma with rather large velocities, which will be accelerated in the electric field, on the average. Such electrons are called *run-away electrons*. The region of velocity space corresponding to them can be called the *region of escape*, and the remaining region can be called the *basic region*. The location of the *boundary of the escape region*, depends on the strength of the electric field E. Due to the Coulomb collisions, electrons from the basic region can fall into the escape region. Part of these electrons, undergoing further collisions, can return to the basic region. Another part is accelerated by the electric field, and enters a regime of continuous acceleration. This process takes place continuously. As a result, the Maxwell velocity distribution which exists at the initial moment changes constantly. A constantly increasing portion of the electrons changes from the basic region to the escape region - i.e., it is drawn into a regime of continuous acceleration. This comprises the phenomenon of electron escape.

2. The location of the boundary of the escape region can be simply found with the aid of equation (13.1). We shall use a rapid plasma electron as the test particle. In contrast to equation (13.1), we shall designate the absolute value of the electron charge by e. The direction of the force acting upon it ($-eE$) will be assumed to be a positive direction of the axis x, and we shall project equation (13.1) on this axis. As a result, we obtain

$$\left\langle \frac{dp_x}{dt} \right\rangle = -\frac{4\pi e^2}{v^3} v_x \sum^* \frac{Le^{*2}}{\mu} n^* \Phi_1 (b^* v) + eE. \qquad (13.2)$$

In the derivation of this equation, it was assumed that all of the electrons, with which the test electron collides, have a Maxwell velocity distribution. In the problem which interests us, the velocity distribution of the electrons is not a Maxwell distribution. However, this is not of great importance, since stages of the process are also examined when the number of electrons in the escape region is small as compared with the number of them in the basic region. In actuality, in this case the collisions of the test electron with the run-away electrons can be disregarded. The electrons in the basic region - with the exception of a small number of them close to the boundary of the escape region - have almost a Maxwell velocity distribution.

In equation (13.2) the averaging is carried out with respect to a group of test electrons characterized by identical values of the vector $\mathbf{v}$. In the problem which we are discussing, we must be concerned with the group which is characterized by *one and the same values of the component* v_x . Therefore, we shall subject expression (13.2) to secondary averaging, namely to averaging with respect to Maxwell distribution of transverse velocities for a fixed value of v_x. For the average in this sense, we obtain /133

$$\left\langle \frac{\Phi_1 (b^* v)}{v^3} \right\rangle = \frac{\int_0^\infty \frac{\Phi_1 (b^* v)}{v^3} e^{-b^{*2} v_\perp^2} v_\perp \, dv_\perp}{\int_0^\infty e^{-b^{*2} v_\perp^2} v_\perp \, dv_\perp} =$$

$$= 2b^{*2} \int_0^\infty \frac{\Phi_1 (b^* v)}{v^3} e^{-b^{*2} v_\perp^2} v_\perp \, dv_\perp,$$

where the transverse velocity component of the electron $v_\perp^2 = v^2 - v_x^2$ is designated by $v_\perp$. For a fixed v_x , we have $v_\perp dv_\perp = v \, dv$. Therefore, introducing a new integration variable $\xi = b^* v$ we obtain

$$\left\langle \frac{\Phi_1 (b^* v)}{v^3} \right\rangle = 2b^{*3} e^{b^{*2} v_x^2} J (b^* v_x). \qquad (13.3)$$

Here the following integral is designated by $J(x)$

$$J (x) = \int_x^\infty \frac{\Phi_1 (\xi)}{\xi^2} e^{-\xi^2} d\xi. \qquad (13.4)$$

As a result, after secondary averaging the equation (13.2) changes into

$$\left\langle\left\langle \frac{dp_x}{dt} \right\rangle\right\rangle = -8\pi e^2 v_x \sum^* \frac{Le^{*2} b^{*2}}{\mu} n^* J (b^* v_x) e^{b^{*2} v_x^2} + eE. \qquad (13.5)$$

If the right part of this equation is negative, then an electron with the fixed value of v_x will be slowed down on the average in the direction of the x-axis; in the opposite case, it will be accelerated. The location of the boundary of the escape region can be determined, if the right part of equation (13.5) is set equal to zero.

3. We are interested in the case when the argument b^*v_x is large. Then it is possible to set $\Phi_1(\xi) = 1$ in the integral (13.4). (For $\xi = 2$, the error which is thus introduced does not exceed 5%, and for $\xi = 3$ it is about 0.03%). Then we have

$$J(x) = \int_x^\infty \frac{e^{-\xi^2}}{\xi^2} d\xi = \frac{e^{-x^2}}{x} - \sqrt{\pi}[1 - \Phi(x)]. \tag{13.6}$$

If the asymptotic series (6.10) is used, then we can write

$$\int_x^\infty \frac{e^{-\xi^2}}{\xi^2} d\xi \to e^{-x^2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1 \cdot 3 \ldots (2k-1)}{2^k \cdot x^{2k+1}}. \tag{13.7}$$

Confining ourselves to the first term of this series, we obtain /134

$$J(x) = \frac{e^{-x^2}}{2x^3}. \tag{13.8}$$

This formula gives somewhat exagerated values for $J(x)$. The error does not exceed $\frac{3}{4x^5} e^{-x^2}$; for $x = 2$, it is about 30%, for $x = 3$ - about 16%, for $x = 4$ - about 10%, etc. Such errors are not significant in approximate calculations which are in the nature of estimates.

After expression (13.8) is substituted in formula (13.5), the latter changes into

$$\left\langle\!\left\langle \frac{dp_x}{dt} \right\rangle\!\right\rangle = -\frac{4\pi e^2}{v_x^2} \sum^* \frac{Le^{*2}n^*}{\mu} + eE. \tag{13.9}$$

Assuming that the ions are singly-charged, and substituting the values of the reduced masses for electron-electron $\left(\mu = \frac{m}{2}\right)$ and electron-ion ($\mu = m$) collisions, we obtain

$$\left\langle\!\left\langle \frac{dp_x}{dt} \right\rangle\!\right\rangle = -\frac{12\pi e^4 Ln}{mv_x^2} + eE \tag{13.10}$$

(n is the electron concentration; m is the electron mass). Collisions with electrons are twice as effective as collisions with ions.

Let us introduce the so-called *critical field*

$$E_{cr} = \frac{4\pi e^3 Ln}{T} = \frac{1}{2}\frac{eL}{D^2}, \tag{13.11}$$

where D is the Debye radius. Then equation (13.10) can be rewritten in the form

$$\left\langle\left\langle \frac{dp_x}{dt} \right\rangle\right\rangle = e\left(E - \frac{3T}{mv_x^2} E_{cr}\right). \tag{13.12}$$

As is clear from its derivation, this equation is applicable when the argument $b^* v_x \equiv \sqrt{\frac{m}{2T}}\, v_x$ for the electrons is large (it is almost sufficient that it be not less than 2 - 3). Setting its right part equal to zero, we find the position of the boundary for the escape region. If

$$E > \frac{3T}{mv_x^2} E_{cr}, \tag{13.13}$$

then the electrons will be continuously accelerated, on the average, in the direction of the x-axis.

4. In order to determine the magnitude of the critical field, let us take a numerical example.
Let $T_e = T_i = 1$ kev $= 1.6 \cdot 10^{-9}$ erg, $n = 10^{15}$ cm^{-3}, and, consequently, $L = 14.1$. Then from (13.11), we obtain

$$E_{cr} = 0{,}012 \text{ cgse} = 3{,}6 \text{ v/cm}.$$

Although this field is small, it is difficult to form it in a fully-ionized plasma due to its high electrical conductivity. Therefore, a study of the escape phenomenon of electrons in *weak fields* - i.e., fields whose strength is much less than the strength of the critical field - is of primary interest. We shall confine ourselves to an examination of this case. /135

In qualitative terms, the picture of electron escape in weak electric fields can be represented as follows. At the moment the electric field is switched on, the velocity distribution of the electrons is strictly a Maxwell distribution. After the electric field is switched on, the electron velocity distribution in the basic region continues to become almost a Maxwell distribution. However, a weak stream of electrons is applied to this distribution in velocity space; this stream carries electrons from the basic region to the escape region. At first it is non-stationary. Then, after a short period of time τ_{est}*, which can be called the *time required to establish the stream*, a *quasi stationary stream* is established, which is proportional to the concentration of electrons in the basic region. Quantitatively, this stream is defined as the number of electrons per unit of plasma volume which leave the basic region each second and go into the escape region. Designating it by S we can write

*Note: est designates 'establish'

$$\frac{dn}{dt} = -S, \tag{13.14}$$

where n is the electron concentration in the basic region. Since the latter is large as compared with the electron concentration in the escape region, the current S is almost constant. The velocity distribution of the electrons in the basic region is slightly distorted by the presence of this stream - this distribution remains almost a Maxwell distribution. The case is somewhat different in the escape region. The number of electrons passing from the basic region to the escape region can be comparable to, and can even surpass, the number of electrons located in this latter region at the beginning moment of time. Therefore, the initial Maxwell velocity distribution in the escape region, and close to its border, is distorted so much that it assumes a completely different character.

5. The time required to establish τ_{est} the quasi stationary stream agrees in order of magnitude with the relaxation time for an individual plasma electron which has the velocity v - i.e., according to equation (12.6) it is determined by the expression

$$\tau_{est} \approx \frac{m^2 v^3}{8\pi e^4 Ln}. \tag{13.15}$$

If the mean thermal velocity v_{therm} of an electron, and the mean electron relaxation time, which corresponds to it, for the plasma as a whole, are introduced, we have

$$\bar{\tau}_e = \frac{m^2 v_{\text{therm}}^3}{8\pi e^4 Ln}, \tag{13.16}$$

then this formula can be written in the following form

$$\tau_{est} \approx \left(\frac{v}{v_{\text{therm}}}\right)^3 \bar{\tau}_e. \tag{13.17}$$

The velocity of an electron at the boundary of the escape region is used as v. At this boundary $\left(\frac{v}{v_{\text{therm}}}\right) \approx \frac{E_{\text{cr}}}{E}$, and we obtain

$$\tau_{est} \approx \left(\frac{E_{cr}}{E}\right)^{3/2} \bar{\tau}_e. \tag{13.18}$$

6. With respect to the stream S, a qualitative approximation of its magnitude contains an element of indeterminate form which is too large [1]. Therefore, it is inadvisable to make this estimation here.

[1] With the aid of different qualitative considerations, it can be readily established that S must contain a multiplier having the form $e^{-\alpha \frac{E}{E_{\text{cr}}}}$, where α is the numerical coefficient on the order of unity. It is impossible to establish a more accurate value for this coefficient from qualitative considerations. This fact makes the qualitative estimates very indefinite.

The calculation of the quantity S must be based on the kinetic equation for a plasma. The effect of run-away electrons has been studied in the works of Dreicer (Ref. 14, Ref. 15) and A. V. Gurevich (Ref. 16, Ref. 17), from the point of view of the theory. There are significant discrepancies between the results obtained in these studies, which we shall not discuss here. For purposes of a general orientation, we shall cite only the results obtained by A. V. Gurevich (Ref. 16). In our notation, we have

$$S = \frac{3}{\bar{\tau}_e} \sqrt{\frac{3}{\pi}} n \left(\frac{E}{E_{cr}}\right)^{1/2} \exp\left(-\frac{E_{cr}}{4E} + \sqrt{\frac{2E_{cr}}{E}}\right), \tag{13.19}$$

where the time $\bar{\tau}_e$ is determined by the expression (12.8). The basic (exponential) multiplier given by Dreicer has the form $e^{\frac{-E_{cr}}{2E}}$.

7. In systems with no electrodes, with a vortex electric field, all of the plasma electrons must take part in the acceleration process due to the presence of the stream S - if only to provide that particles do not depart at the wall. The idea of creating a *high-current electron accelerator* was based on this fact; it was first presented by Steinbeck. The time t_a, which is necessary for the basic mass of plasma electrons to enter into a regime of continuous acceleration for a given value of the strength of the electrical field E, is determined in order of magnitude by the expression

$$t_a = \frac{n}{S}. \tag{13.20}$$

However, the possibility of the existence of a mechanism representing an increase in the intensity of the electron streams is not excluded. This mechanism could consist of *longitudinal plasma fluctuations*, /137 which are excited by a bundle of rapid electrons (Ref. 16). In their turn, these fluctuations act upon the bundle which has excited them, and can lead to *an anomalous retardation of it* (Ref. 18).

14. FOKKER-PLANCK EQUATION

1. When a plasma is not in a state of thermodynamic equilibrium, a rigorous examination of the different types of processes taking place in it is possible, generally speaking, only on the basis of the *kinetic equation*. In the derivations of this equation, we shall confine ourselves to examining only a *fully-ionized* or *hot plasma*, in which the processes of *ionization, recombination, and excitation* of the particles play hardly any role. This means that our results refer only to a fully-ionized plasma, consisting of electrons and *bare* atomic nuclei. The internal state of each plasma particle does not change. From a classical point of view, the particle can be regarded

as a *material particle of classical mechanics*. The phase space of such a particle is six-dimensional space, each point of which is characterized by three rectangular coordinates x_1, x_2, x_3, which determine the position of the particles in normal three-dimensional space, and the impulses which correspond to them p_1, p_2 p_3. It can also be stated that the position of the point in phase space is determined by the *six-dimensional vector* $\vec{\xi}$, with the coordinates

$$\begin{aligned} \xi_1 &= x_1, \quad \xi_2 = x_2, \quad \xi_3 = x_3, \\ \xi_4 &= p_1, \quad \xi_5 = p_2, \quad \xi_6 = p_3. \end{aligned} \tag{14.1}$$

Let us assume that at the moment of time under consideration the particle is located at the point $\vec{\xi}$ of phase space, if it has the coordinates (14.1) at this moment of time.

The concentration $F(t, \vec{\xi})$ *of particles in phase space can be called their distribution function.* In terms of this definition, the number dN of particles, which are located at the moment of time t in the volume element $d\vec{\xi} \equiv d\xi_1 d\xi_2 \ldots d\xi_6$ of phase space with the center at the point $\vec{\xi}$, is related to the distribution function by the relationship

$$dN = F\left(t, \vec{\xi}\right) d\vec{\xi}. \tag{14.2}$$

It is understood that here - as in all statistical studies - we are not discussing the true number of particles dN in the element of phase volume $d\vec{\xi}$, but rather its *smoothed out* value with respect to the volumes of phase space which are *infinitely small in physical terms.*

For a complete statistical description of the plasma, it is necessary to introduce as many distribution functions as there are types of particles. A particle of each type will have its corresponding phase space. From this point on, the distribution function for the particles of any definite (but arbitrary) type will be designated by $F(t, \vec{\xi})$. /138

Instead of phase space (i.e., coordinate impulse space), it is sometimes more convenient to use *coordinate-velocity space* with the volume element $d\mathrm{x}\, d\mathrm{v} \equiv dx_1 dx_2 dx_3 dv_1 dv_2 dv_3$. The distribution function in this space will be designated by the small letters $f(t, \mathrm{x}, \mathrm{v})$. Then, instead of the relationship (14.2), we can write

$$dN = f(t, \mathrm{x}, \mathrm{v})\, d\mathrm{x}\, d\mathrm{v}. \tag{14.3}$$

The equation or system of equations, which describe the change in the distribution functions in time and in phase space (or coordinate-velocity space), can be called a kinetic equation. In the case when

there are no ionization and recombination processes (in the more general sense - the processes by which the particles are interchanged with each other), the kinetic equation expresses the *conservation of a number of particles* of each type.

2. We can obtain the kinetic equation by the same method which is used in hydrodynamics, in deriving the *equation of continuity* in the Euler representation. Let us examine a fixed phase volume Ω, which is bounded by a closed five-dimensional surface Σ. The particles move in phase space, and this motion can be described by the vector $\vec{\xi}$ as *six-dimensional velocity*. A portion of the particles leaves the volume Ω through the surface Σ; the other portion enters it from the surrounding sections of phase space. For this reason, and only for this reason, the number of particles within the volume Ω changes. Generally speaking, the vector $\vec{\xi}$ of each particle changes constantly in time. However, when two or more particles converge toward each other, very strong forces develop between them, and the velocity $\mathbf{v}$ of each particle changes sharply throughout very short time intervals. In an idealized picture, these times are disregarded, and the velocity change is regarded as an instantaneous process - collision. With such an idealization, the vector $\mathbf{v}$ and the vector $\vec{\xi}$ along with it, experience a discontinuity at the time of collision. This means that, as a result of the collision, the particle crosses instantaneously from one point of phase space to another. The collisions can be divided into *close* and *far-removed* collisions. For close collisions, the vector $\mathbf{v}$ (and, consequently, the vector $\vec{\xi}$) changes greatly, while for far-removed collisions - it changes very little. As a rule, the far-removed collisions play a more significant role than the close collisions, since the latter are relatively rare. This is verified by the large values of the Coulomb logarithms, by means of which the far-removed collisions can be taken into consideration. If the close collisions are completely disregarded - taking into account only the far-removed collisions, then only the small particle jumps stop in phase space. This jump-like motion can be approximated fairly accurately by level, smooth motion. Therefore, the motion of a set of particles in phase space can be described by *the six-dimensional vector* $\vec{J}^{6}$ *of the particle stream density*, similarly to the manner in which the motion of a liquid is described in hydrodynamics. With respect to the definition of the vector $\vec{J}^{6}$, the number of particles which pass through an elementary, five-dimensional surface $d\Sigma$ each second can be represented in the form $J^{6}_{\nu}\, d\Sigma$, where J^{6}_{ν} is the projection of the vector $\vec{J}^{6}$ in the direction $\vec{\nu}$ normal to the surface $d\Sigma$. Thus, the projection of the vector $\vec{J}^{6}$ is determined in any direction in phase space, and consequently the vector itself $\vec{J}^{6}$. With the aid of the

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vector $\vec{J}^{6}$, the condition for the conservation of a number of particles can be described in the form of a *six-dimensional equation of continuity*

$$\frac{\partial F}{\partial t}+\frac{\partial J_{\alpha}^{(6)}}{\partial \xi_{\alpha}}=0, \tag{14.4}$$

where in accordance with the tensor symbolics, it is understood that the summation $(\alpha = 1.2, \ldots, 6)$ is made with respect to the coordinate index α which occurs twice. The problem is reduced to determining the six-dimensional vector $\vec{J}^{6}$.

3. In order to solve this problem, let us conceive of a small five-dimensional surface $d\Sigma$ in six-dimensional phase space $\vec{\xi}(x, p)$; this surface is perpendicular to some coordinate axis ξ_{α} of this space $(\alpha = 1.2, \ldots, 6)$. We shall assume that the positive direction of the axis ξ_{α} is the positive direction of the normal to the surface $d\Sigma$. Let us determine the excess dN of a number of the particles belonging to the type under consideration, which pass through the surface $d\Sigma$ in the positive direction during a small time interval τ, as compared to the number of particles passing in the opposite direction.

Among the particles of the type under consideration, let us conceive of a fairly large group of particles which are displaced in phase space by the same six-dimensional vector $\Delta\vec{\xi}^{i}$ during the time τ. We shall use $F^{i}(t, \vec{\xi})$ to designate the concentration of the particles in this group in phase space.

The number dN^{i} of particles in the group under consideration, which pass through the surface $d\Sigma$ in time τ, obviously equals the number of them within an oblique six-dimensional cylinder with the base $d\Sigma$ on the generatrices $\Delta\vec{\xi}^{i}$ (Figure 8) - i.e., this number equals the integral $\int F^{i}(t, \vec{\xi})d\vec{\xi}$ which is taken over the entire volume of this cylinder. Since it can be assumed that the surface $d\Sigma$ is infinitely small, the element of the *phase volume* can be represented /140
in the form $d\vec{\xi} = |d\Sigma \cdot d\xi_{\alpha}|$. In addition, let us designate the radius-vector of the center of the surface $d\Sigma$ by $\vec{\xi}_{0}$. Then, assuming that $\vec{\xi} = \vec{\xi}_{0} + \vec{\varepsilon}$ and - for the sake of brevity - omitting the argument t in the function $F^{i}(t, \vec{\xi})$, we can write

$$dN^{i}=\int_{-\Delta\xi_{\alpha}^{i}}^{0} F^{i}\left(\vec{\xi}_{0}+\vec{\varepsilon}\right) d\Sigma \cdot d\varepsilon_{\alpha}.$$

Let us assume that the time τ is chosen so small that the displacements $\Delta\vec{\xi}^{i}$ are small. Then, expanding $F^{i}(\vec{\xi}_{0} + \vec{\varepsilon})$ in powers of $\vec{\varepsilon}$, and truncating the expansion at the linear terms, we obtain

$$dN^{i}=d\Sigma \cdot F^{i}\left(\vec{\xi}_{0}\right) \int_{-\Delta\xi_{\alpha}^{i}}^{0} d\varepsilon_{\alpha}+d\Sigma \cdot \frac{\partial F^{i}}{\partial \xi_{\beta}} \int_{-\Delta\xi_{\alpha}^{i}}^{0} \varepsilon_{\beta} d\varepsilon_{\alpha}.$$

Integration can be readily carried out, if it is taken into consideration

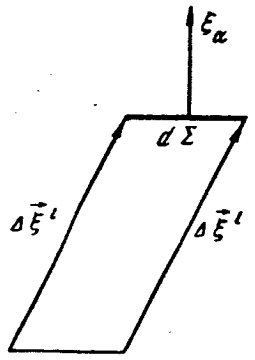

Figure 8.

that all ε_β are proportional to ε_α, due to the assumption regarding the finite smallness of the surface $d\Sigma$.

On this basis, we have

$$dN^i = d\Sigma \cdot \left\{ \Delta\xi^i_\alpha \cdot F^i - \frac{1}{2} \Delta\xi^i_\alpha \Delta\xi^i_\beta \frac{\partial F^i}{\partial \xi_\beta} \right\}.$$

The argument $\vec{\xi}_0$ of the function F^i is omitted here, since from this point on it is assumed that the values of the function F^i and all of its derivatives refer to the center of the surface $d\Sigma$.

The excess dN of the number of particles belonging to the type under consideration, which pass through the surface $d\Sigma$ in a positive direction, as compared to the number of particles going in the opposite direction, can be determined by the summation of the preceding expression over all the values of i - i.e., over all of the groups of particles. According to the definition of the mean, we have

$$\left.\begin{aligned} \sum_i \Delta\xi^i_\alpha \cdot F^i\left(t, \vec{\xi}\right) &= \langle \Delta\xi_\alpha \rangle \cdot F\left(t, \vec{\xi}\right), \\ \sum_i \Delta\xi^i_\alpha \Delta\xi^i_\beta \cdot F^i\left(t, \vec{\xi}\right) &= \langle \Delta\xi_\alpha \Delta\xi_\beta \rangle \cdot F\left(t, \vec{\xi}\right). \end{aligned}\right\} \qquad (14.5)$$

It is obvious that the quantities $\Delta\xi^i_\alpha$, as independent variables, do not depend on $\vec{\xi}$. However, their mean - and also the mean of $\Delta\xi^i_\alpha \Delta\xi^i_\beta$ - generally speaking, depend on $\vec{\xi}$, since F^i and F are functions of $\vec{\xi}$. Therefore, differentiating the second equation with respect to ξ_β, and making a summation over β, we find /141

$$\sum_i \Delta\xi^i_\alpha \Delta\xi^i_\beta \frac{\partial F^i\left(t, \vec{\xi}\right)}{\partial \xi_\beta} = F\left(t, \vec{\xi}\right) \frac{\partial}{\partial \xi_\beta} \langle \Delta\xi_\alpha \Delta\xi_\beta \rangle + \langle \Delta\xi_\alpha \Delta\xi_\beta \rangle \frac{\partial F\left(t, \vec{\xi}\right)}{\partial \xi_\beta}.$$

Utilizing this expression, and introducing the notation

$$J_\alpha^{(6)} = \frac{dN}{d\Sigma}, \tag{14.6}$$

we obtain

$$J_\alpha^{(6)} = \frac{1}{\tau}\left\{\langle \Delta\xi_\alpha\rangle - \frac{1}{2}\frac{\partial}{\partial\xi_\beta}\langle \Delta\xi_\alpha \Delta\xi_\beta\rangle\right\} F - \frac{1}{2\tau}\langle \Delta\xi_\alpha \Delta\xi_\beta\rangle \frac{\partial F}{\partial\xi_\beta}. \tag{14.7}$$

If the following expansion terms are taken into consideration in an expansion of the function $F^i(\vec{\xi}_0 + \vec{\varepsilon})$, then we would obtain the following expression, instead of expression (14.7):

$$J_\alpha^{(6)} = \frac{1}{\tau}\left\{\langle \Delta\xi_\alpha\rangle - \frac{1}{2}\frac{\partial}{\partial\xi_\beta}\langle \Delta\xi_\alpha \Delta\xi_\beta\rangle + \frac{1}{6}\frac{\partial^2}{\partial\xi_\beta\partial\xi_\gamma}\langle \Delta\xi_\alpha \Delta\xi_\beta \Delta\xi_\gamma\rangle\right\} F -$$
$$-\frac{1}{2\tau}\left\{\langle \Delta\xi_\alpha \Delta\xi_\beta\rangle - \frac{2}{3}\frac{\partial}{\partial\xi}\langle \Delta\xi_\alpha \Delta\xi_\beta \Delta\xi_\gamma\rangle\right\}\frac{\partial F}{\partial\xi_\beta} +$$
$$+\frac{1}{6\tau}\langle \Delta\xi_\alpha \Delta\xi_\beta \Delta\xi_\gamma\rangle \frac{\partial^2 F}{\partial\xi_\beta\partial\xi_\gamma}. \tag{14.8}$$

However, we shall confine ourselves to the simpler expression (14.7). Formula (14.8) can be used to estimate the error in the approximation being used.

The averaging in the tensors $\langle \Delta\xi_\alpha\rangle$, $\langle \Delta\xi_\alpha\Delta\xi_\beta\rangle$, ... is carried out with respect to the groups of particles which are determined in such a way that, during the time τ, the particles of each group undergo one and the same displacement $\Delta\vec{\xi}^i$ in phase space. These groups can be selected in such a way that the concentration of particles in each group in normal space is small; then the interaction between the particles of one and the same group can be disregarded. In this approximation, the particles of the same group behave like *independent* particles. Therefore, in calculating the mean values of $\langle \Delta\xi_\alpha\rangle$ and $\langle \Delta\xi_\alpha\Delta\xi_\beta\rangle$, it can be assumed that each group consists of only one particle. Thus, the averaging operation is reduced to the averaging over all of the particles of the type under consideration in the vicinity of the phase point $\vec{\xi}(x, p)$.

4. If the time τ is rather small, then - within an accuracy of terms on the order of τ^2 - the increases in the coordinates Δx_α, during the time τ, can be represented in the form $\Delta x_\alpha = v_\alpha\tau$, where v is the velocity vector of a particle in normal three-dimensional space. As regards the increases in the impulses, they can be divided into two parts: $\Delta p = \Delta' p + \Delta'' p$. Here $\Delta' p$ is the increase caused by the regular field force (for example, electric or magnetic), and such a field is called a self-consistent field. $\Delta'' p$ is the increase caused by the collisions of the particle under consideration with other particles. Within an accuracy of the square of τ, we can write $\Delta' p_\alpha = X_\alpha\tau$ for $\Delta' p$. Here, X is the force acting on a particle from the self-consistent field. Thus, we have /142

$$\langle \Delta\xi_\alpha \rangle = v_\alpha \tau \qquad (\alpha = 1, 2, 3),$$
$$\langle \Delta\xi_\alpha \rangle = X_\alpha \tau + \langle \Delta''\xi_\alpha \rangle \qquad (\alpha = 4, 5, 6).$$

We shall consider only the far-removed collisions, i.e. those for which the increases $\Delta''\mathbf{p}$ are small as compared with $\mathbf{p}$. Nevertheless, for rather small τ, these increases are large as compared with the increases $\Delta'\mathbf{p}$ caused by the regular force $\mathbf{X}$, since very large (in mathematical idealization - infinitely large) forces of interaction develop during the collision. The directions of the vectors $\Delta''\mathbf{p}$ for the different particles in the vicinity of the phase point $\vec{\xi}(\mathbf{x}, \mathbf{p})$ can be any directions at all. However, these directions are not equally probable, due to the presence of the ordered velocity $\mathbf{v}$ in these particles. Therefore, it can be assumed (this will be demonstrated in the following section for a plasma in the pair collision approach) that, when the quantities $\Delta'' p_\alpha$ are averaged, only the large components disappear, but the small ones remain, which - within an accuracy of terms on the order of τ^2 - are proportional to τ. Thus, we have

$$\langle \Delta'' p_\alpha \rangle = A''_\alpha \tau \qquad (\alpha = 1, 2, 3)$$

or

$$\langle \Delta'' \xi_\alpha \rangle = A''_\alpha \tau \qquad (\alpha = 4, 5, 6),$$

where the vector $\mathbf{A}''$ does not depend on τ.

Let us now turn to an examination of the tensor $\langle \Delta\xi_\alpha \Delta\xi_\beta \rangle$. The increases of the coordinates and the regular increases of the impulses can be disregarded, since these increases are proportional to τ, and the terms of the tensor $\Delta\xi_\alpha \Delta\xi_\beta$, which correspond to them, will be proportional to τ^2 (in our approach, such terms can be disregarded). Only terms of the series $\Delta'' p_\alpha \Delta'' p_\beta$ remain, i.e., terms resulting from the collisions. Under determined conditions, these terms are proportional to the time τ. We can write

$$\Delta''\mathbf{p} = \sum_i \delta^i \mathbf{p},$$

where $\delta'\mathbf{p}$ is the impulse increase in the particle under consideration, as the result of a single (ith) act of collision between it and another particle. Therefore, we have

$$\langle \Delta'' p_\alpha \cdot \Delta'' p_\beta \rangle = \sum_i \sum_j \langle \delta p_\alpha^i \delta p_\beta^j \rangle.$$

If the subsequent collisions between the particle and other particles are *statistically independent*, i.e.

$$\langle \delta p_\alpha^i \delta p_\beta^j \rangle = 0 \qquad \text{for} \quad i \neq j,$$

then

$$\langle \Delta'' p_\alpha \cdot \Delta'' p_\beta \rangle = \sum_i \langle \delta p_\alpha^i \delta p_\beta^i \rangle = z \langle \delta p_\alpha \delta p_\beta \rangle,$$

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where z is the number of collisions for the particle under consideration during the time τ. It is proportional to τ, and our statements are substantiated.

Thus, we arrive at the conclusion that a six-dimensional vector (14.7) - within an accuracy of terms on the order of τ - does not depend on τ, and can be regarded as a vector for the stream density of particles in six-dimensional phase space. With the aid of this vector, the kinetic equation can be written in the form of a six-dimensional equation of continuity (14.4). It can also be written in a three-dimensional form:

$$\frac{\partial F}{\partial t} + v_\alpha \frac{\partial F}{\partial x_\alpha} + \frac{\partial}{\partial p_\alpha}(X_\alpha F) + \frac{\partial I_\alpha}{\partial p_\alpha} = 0, \tag{14.9}$$

where $\alpha = 1, 2, 3$, I is the three-dimensional vector which represents the projection in space of the impulses from the part of the six-dimensional vector $\vec{J}^{\,6}$, which is caused by particle collisions. Vector I gives the *stream density of particles in impulse space* caused by the collisions between particles. It is determined by the expression

$$I_\alpha = A_\alpha F - D_{\alpha\beta}\frac{\partial F}{\partial p_\beta}, \tag{14.10}$$

where

$$A_\alpha = \frac{1}{\tau}\left\{\langle \Delta p_\alpha \rangle - \frac{1}{2}\frac{\partial}{\partial p_\beta}\langle \Delta p_\alpha \Delta p_\beta \rangle\right\}, \tag{14.11}$$

$$D_{\alpha\beta} = \frac{1}{2\tau}\langle \Delta p_\alpha \Delta p_\beta \rangle. \tag{14.12}$$

Here and from this point on, Δp designates only that part of the particle impulse increase which is caused by *collisions* (in the derivation, it was temporarily designated by $\tilde{\Delta} p$), and not the total increase in the particle impulse.

The kinetic equation in the form of (14.9) is called the *Fokker-Planck equation.* The term $D_{\alpha\beta}\frac{\partial F}{\partial p_\beta}$ is called the *diffusion stream of particles in impulse space*, since it is related to the gradient of particle concentration in this space. By analogy with normal diffusion, the tensor $D_{\alpha\beta}$ can be called *the tensor of diffusion in impulse* space. The presence of the term $A_\alpha F$ can be explained by the fact that uniform distribution of the particles with respect to impulse space is not an equilibrium distribution. If this distribution is formed artificially, after a certain amount of time has elapsed it changes by itself into Maxwell-Boltzmann equilibrium distribution. The vector A is called the *coefficient of dynamic*

friction in impulse space.

5. The Fokker-Planck equation can also be written for the distribution function $f(t,\mathbf{r},\mathbf{v})$ in velocity space. It takes the following form /144

$$\frac{\partial f}{\partial t} + v_\alpha \frac{\partial f}{\partial x_\alpha} + \frac{1}{m}\frac{\partial}{\partial v_\alpha}(X_\alpha f) + \frac{\partial j_\alpha}{\partial v_\alpha}. \tag{14.13}$$

Here, j is the *three-dimensional vector of particle stream density in velocity space* resulting from the collisions between particles. It is determined by the expression

$$j_\alpha = a_\alpha f - d_{\alpha\beta}\frac{\partial f}{\partial v_\beta}, \tag{14.14}$$

where

$$a_\alpha = \frac{1}{\tau}\left\{\langle \Delta v_\alpha\rangle - \frac{1}{2}\frac{\partial}{\partial v_\beta}\langle \Delta v_\alpha \Delta v_\beta\rangle\right\}, \tag{14.15}$$

$$d_{\alpha\beta} = \frac{1}{2\tau}\langle \Delta v_\alpha \Delta v_\beta\rangle. \tag{14.16}$$

The tensor $d_{\alpha\beta}$ is called the *diffusion tensor in velocity space*, and a_α (according to Chandrasekhar) is the *coefficient of dynamic friction.* It is obvious that

$$\begin{aligned} A_\alpha &= ma_\alpha, \\ D_{\alpha\beta} &= m^2 d_{\alpha\beta}. \end{aligned} \tag{14.17}$$

15. THE RELATIONSHIP BETWEEN THE DIFFUSION TENSOR AND THE DYNAMIC FRICTION COEFFICIENT, AND THE DISTRIBUTION FUNCTION. THE KINETIC EQUATION IN THE LANDAU FORM.

1. In order to obtain the kinetic equation of a plasma in final form from the Fokker-Planck equation, it is necessary to express the mean values of $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$ by the distribution function of plasma particles. We should note that Δp here designates the impulse change in the particle under consideration, which it undergoes during the time τ as a result of collisions with other plasma particles. In solving this problem, we shall use the customary assumption of the theory of pair collisions. According to this theory, in the calculation of a similar type of mean values it can be assumed that the *total impulse change* Δp *equals the vector sum* $\Sigma \delta p$ *of those changes which a particle impulse would undergo during subsequent collisions, which were independent of each other, with other plasma particles.*

Let us first examine the collisions of the particle under consideration with only a group of uniform particles which have one and the same impulse $\mathbf{p}^*$, with scattering within the small element $d\mathbf{p}^*$. The concentration of such particles in normal space equals $F^*(\mathbf{p}^*)d\mathbf{p}^*$. Then the relative velocity $\mathbf{u} = \mathbf{v} - \mathbf{v}^*$ can be assumed to be the same for collisions with all particles in the group under consideration.

Let us first conceive of a special coordinate system whose z-axis is directed along the relative velocity $\mathbf{u}$, and the small x- and y-axes are perpendicular to it. Let us calculate the mean value of $\langle \Delta_1 p_\alpha \rangle$ and $\langle \Delta_1 p_\alpha \cdot \Delta_1 p_\beta \rangle$, where $\Delta_1 \mathbf{p}$ is the impulse change in the /145

particle under consideration (test particle) during the time τ, caused by the collisions between it and the particles in the group under consideration. Since the azimuth of the collision plane can assume the values from 0 to 2π with equal probability, only the mean values of $\langle \Delta_1 p_z \rangle$ $\langle \Delta_1 p_z^2 \rangle$ and $\langle \Delta_1 p_y^2 \rangle$ will differ from zero. (The quantity $\langle \Delta_1 p_z^2 \rangle$ must be assumed to equal zero and be infinitely small of a higher order in the given approximation). For the same reason $\langle \Delta_1 p^2 \rangle = \langle \Delta_1 p_y^2 \rangle = \frac{1}{2} \langle \Delta_1 p_\perp^2 \rangle$, where $\Delta_1 p_\perp$ is the change in the component of the impulse $\mathbf{p}$, which is perpendicular to the relative velocity $\mathbf{u}$. Thus, in formula (2.16), we have

$$\langle \Delta_1 p_z \rangle = -\frac{1}{2u\mu} \langle \Delta_1 p_\perp^2 \rangle, \tag{15.1}$$

so that it is sufficient to calculate only $\langle \Delta_1 p_z \rangle$. We can determine the latter quantity by summation $\delta p_z \equiv \delta p_\parallel = -2\mu u \sin^2 \frac{\vartheta}{2}$ [see formula (2.12)] over all the collisions with the particles of the group under consideration during the time τ. Changing from summation to integration, we can write

$$\langle \Delta_1 p_z \rangle = F^*(\mathbf{p}^*)\, u d\mathbf{p}^* \tau \int \delta p_z \sigma(\vartheta, u)\, d\Omega.$$

Substituting expression (3.5) for $\sigma(\vartheta, u)$ and truncating the lower limit of the integral in the usual way, we obtain

$$\langle \Delta_1 p_z \rangle = -\frac{4\pi (ee^*)^2}{\mu u^2} \tau L F^*(\mathbf{p}^*)\, d\mathbf{p}^*,$$

where L is the Coulomb logarithm. On the basis of formula (15.1), we find

$$\langle \Delta_1 p_x^2 \rangle = \langle \Delta_1 p_y^2 \rangle = \frac{1}{2} \langle \Delta p_\perp^2 \rangle = \frac{4\pi (ee^*)^2}{u} \tau L F^*(\mathbf{p}^*) d\mathbf{p}^*.$$

Thus, in the coordinate system under consideration the tensor $\langle \Delta_1 p_\alpha \Delta_1 p_\beta \rangle$

has a diagonal form, while $\langle \Delta_1 p_z^2 \rangle = 0$ in the approximation being used. The vector $\langle \Delta_1 \mathbf{p} \rangle$ contains only one component which is different from zero (the z-component).

If we change to an arbitrary coordinate system this tensor and this vector can be written in the form

$$\langle \Delta_1 p_\alpha \Delta_1 p_\beta \rangle = 4\pi (ee^*)^2 \tau L u_{\alpha\beta} F^* (\mathbf{p}^*)\, d\mathbf{p}^*, \tag{15.2}$$

$$\langle \Delta_1 p_\alpha \rangle = -\frac{4\pi (ee^*)^2}{\mu u^3} \tau L u_\alpha F^* (\mathbf{p}^*)\, d\mathbf{p}^*, \tag{15.3}$$

where the following tensor is designated by $u_{\alpha\beta}$:

$$u_{\alpha\beta} = \frac{u^2 \delta_{\alpha\beta} - u_\alpha u_\beta}{u^3} = \frac{\partial^2 u}{\partial u_\alpha \partial u_\beta}. \tag{15.4}$$

All that remains now is to integrate expressions (15.2) and (15.3) with respect to $d\mathbf{p}^*$ and to make a summation over all types of field particles. As a result, we obtain /146

$$D_{\alpha\beta} = \sum^* 2\pi L (ee^*)^2 \int u_{\alpha\beta} F^* (\mathbf{p}^*)\, d\mathbf{p}^*, \tag{15.5}$$

$$A_\alpha = -\sum^* 2\pi L (ee^*)^2 \int \left(\frac{\partial u_{\alpha\beta}}{\partial p_\beta} + \frac{2u_\alpha}{\mu u^3} \right) F^* (\mathbf{p}^*)\, d\mathbf{p}^*. \tag{15.6}$$

The formulas obtained provide a solution for the problem which was posed at the beginning of the present section.

2. The expression for A_α can be transformed to a more symmetrical form. First of all, we have

$$\frac{\partial u_{\alpha\beta}}{\partial p_\beta} = \frac{1}{m} \frac{\partial u_{\alpha\beta}}{\partial v_\beta} = \frac{1}{m} \frac{\partial u_{\alpha\beta}}{\partial u_\beta} = \frac{1}{m} \frac{\partial}{\partial u_\alpha} \frac{\partial^2 u}{\partial u_\beta \partial u_\beta} = -\frac{2}{m} \frac{u_\alpha}{u^3}. \tag{15.7}$$

In addition,

$$\frac{\partial u_{\alpha\beta}}{\partial p_\beta} + \frac{2u_\alpha}{\mu u^3} = \frac{2u_\alpha}{u^3} \left(\frac{1}{\mu} - \frac{1}{m} \right) = \frac{2u_\alpha}{m^* u^3}.$$

Consequently,

$$A_\alpha = -\sum^* 4\pi (ee^*)^2 L \int \frac{u_\alpha}{m^* u^3} F^* (\mathbf{p}^*)\, d\mathbf{p}^*. \tag{15.8}$$

If the velocity of a field particle with respect to a test particle is designated by $\mathbf{u}^*$, then $\mathbf{u}^* = -\mathbf{u}$, and on the basis of the relationship (15.7), we can write

$$\frac{\partial u_{\alpha\beta}}{\partial p^*_\beta} = -\frac{2}{m^*}\frac{u^*_\alpha}{u^{*3}}$$

or

$$\frac{\partial u_{\alpha\beta}}{\partial p^*_\beta} = \frac{2}{m^*}\frac{u_\alpha}{u^3}.$$

Therefore formula (15.8) is changed into

$$A_\alpha = -\sum^* 2\pi (ee^*)^2 L \int \frac{\partial u_{\alpha\beta}}{\partial p^*_\beta} F^*(\mathbf{p}^*)\, d\mathbf{p}^*.$$

After integration by parts, we obtain

$$A_\alpha = \sum^* 2\pi (ee^*)^2 L \int u_{\alpha\beta} \frac{\partial F^*(\mathbf{p}^*)}{\partial p^*_\beta} d\mathbf{p}^*. \qquad (15.9)$$

Finally, formulas (14.10), (15.5) and (15.9) give

$$I_\alpha = \sum^* 2\pi (ee^*)^2 L \int\left\{ F(\mathbf{p}) \frac{\partial F^*(\mathbf{p}^*)}{\partial p^*_\beta} - F^*(\mathbf{p}^*) \frac{\partial F(\mathbf{p})}{\partial p_\beta}\right\} u_{\alpha\beta} d\mathbf{p}^*. \qquad (15.10)$$

This is the *collision integral* which was first obtained by Landau (Ref. 10, Ref. 11) by a different, more formal method. Introducing it into equation (14.9), we obtain *a kinetic equation in the Landau form.*

The Landau equation can also be written in the form (14.13), /147
utilizing velocity space instead of impulse space. On the basis of formulas (14.14) and (14.17), the vector $\mathbf{j}$ will be determined by the expression

$$\mathbf{j}_\alpha = \sum^* \frac{2\pi (ee^*)^2 L}{m} \int\left\{ \frac{f(\mathbf{v})}{m^*} \cdot \frac{\partial f^*(\mathbf{v}^*)}{\partial v^*_\beta} - \frac{f^*(\mathbf{v}^*)}{m} \frac{\partial f(\mathbf{v})}{\partial v_\beta}\right\} u_{\alpha\beta} d\mathbf{v}^*. \qquad (15.11)$$

3. For certain purposes, it is advisable to transform the expressions for the vector $\mathbf{A}$ and the tensor $D_{\alpha\beta}$ into another form, which is particularly suitable for calculating these quantities in terms of the given distribution function. For this purpose, let us change from integration with respect to impulse space to integration with respect to velocity space. In addition, let us take the fact into account that

$$u_{\alpha\beta} = \frac{\partial^2 u}{\partial v_\alpha \partial v_\beta}. \qquad (15.12)$$

Then formulas (15.8) and (15.5) can be rewritten in the following form:

$$\mathbf{A} = -\sum^* \frac{4\pi (ee^*)^2 L}{m^*} \int \frac{\mathbf{u}}{u^3} f^*(\mathbf{v}^*)\, d\mathbf{v}^*, \qquad (15.13)$$

$$D_{\alpha\beta} = \sum^* 2\pi (ee^*)^2 L \frac{\partial^2}{\partial v_\alpha \partial v_\beta} \int u f^*(\mathbf{v}^*)\, d\mathbf{v}^* \qquad (15.14)$$

or more concisely

$$\mathbf{A} = -\sum^{*} \frac{L\mathbf{E}_v}{m^*}, \tag{15.15}$$

$$D_{\alpha\beta} = \sum^{*} L \frac{\partial^2 \Psi(\mathbf{v})}{\partial v_\alpha \partial v_\beta}. \tag{15.16}$$

Here $\mathbf{E}_v$ is the analog of the electrostatic field in velocity space, which was previously introduced in Section 3. We should note that it is determined by the expression

$$\mathbf{E}_v = \int \frac{\mathbf{u}}{u^3} \varrho_v(\mathbf{v}^*)\, d\mathbf{v}^*, \tag{15.17}$$

where

$$\varrho_v = 4\pi (ee^*)^2 f^*(\mathbf{v}^*). \tag{15.18}$$

By definition, the function $\Psi(\mathbf{v})$ equals

$$\Psi(\mathbf{v}) = \frac{1}{2} \int u \varrho_v(\mathbf{v}^*)\, d\mathbf{v}^*. \tag{15.19}$$

Functions of this type were first introduced by Rosenbluth, MacDonald and Judd (Ref. 2), and by B. A. Trubnikov (Ref. 1) independently of them.

4. Formulas (15.10), and (15.11), and all of the expressions equivalent to them, were introduced under the assumption that *the effect of the force field on the particle collisions can be disregarded.* This is not always admissible in the force fields, due to the finiteness of the time required for particle collisions. The case when we are dealing with a magnetic force field is of practical importance. Then, the *expressions (15.10) and (15.11) can be used,* /148 *if the Debye radius (4.4) is small as compared with the Larmor radius*

$$r_1 = \frac{mcv}{eB} \approx \frac{c}{eB} \sqrt{3mT}. \tag{15.20}$$

The Coulomb force of attraction or repulsion of the interacting particles is screened at distances on the order of D. Therefore, in order that the formulas (15.10) and (15.11) may be applicable, it is sufficient that at distances on this order the magnetic field B distorts the particle trajectory slightly. This is possible if the following condition is fulfilled

$$\frac{D}{r_1} \approx \frac{B}{2c\sqrt{6\pi nm}} \ll 1, \tag{15.21}$$

where n is the particle concentration of one sign (the plasma is

assumed to be quasi-neutral). For electrons:

$$\frac{D}{r_1} \approx \frac{127B}{\sqrt{n}},$$

for hydrogen ions

$$\frac{D}{r_1} \approx \frac{3B}{\sqrt{n}},$$

for deuterium ions

$$\frac{D}{r_1} \approx \frac{2{,}1B}{\sqrt{n}},$$

where B is given in gauss.

It is interesting to determine whether condition (15.21) is fulfilled in a different type of *magnetic traps*, which have been developed in connection with a study of the problem of *controlled thermonuclear reactions*. For this purpose, let us give condition (15.21) a somewhat different form. Let us introduce the generally-accepted designation

$$\beta = \frac{8\pi p}{B^2} \qquad (15.22)$$

for the relationship of the plasma pressure p to the magnetic pressure $\frac{B^2}{8\pi}$. We shall thus examine the pressures of the electron and ion plasma components separately, so that the latter will be characterized by two values of β - one for electrons and the other for ions. The quantity β can not be greater than unity, since otherwise the magnetic pressure could not counterbalance the pressure of the plasma particles - i.e., it could not hold it in the trap. Since $p = nT$, condition (15.21) can be readily transformed to the form

$$\beta \gg \frac{1}{3\pi} \frac{T}{mc^2}. \qquad (15.23)$$

If we tentatively assume $T = 50$ kev then for deuterons /149
$\frac{T}{mc^2} = \frac{1}{40\ 000}$, and condition (15.23) requires that $\beta >> \frac{1}{400\ 000}$, which is fulfilled very well in our arrangements. However, for electrons $\frac{T}{mc^2} = \frac{1}{10}$, and this condition changes into $\beta >> \frac{1}{100}$. In certain arrangements - for example, in stellarators, it is either not fulfilled or fulfilled very poorly. In these cases, formulas (15.10) and (15.11) provide correct results for only that part of the collision integral which is caused by the scattering of ions by ions. For collision integrals, connected with the scattering of ions by electrons or with the scattering of electrons by ions and

electrons, these formulas can at most be used only to estimate the order of magnitude.

5. It is obvious that the kinetic equation in the Landau form is approximate. We disregarded the effect of far-removed collisions in deriving it. It contains a poorly-determined quantity - the Coulomb logarithm L. The value of the latter depends, in particular, on the plasma temperature - a quantity which has a clear and distinct meaning only for thermodynamically equilibrium states. An explicit kinetic equation, which is used to describe the non-equilibrium states and processes in a plasma, cannot contain such quantities as the plasma temperature as parameters. Therefore, *the Landau equations can be used only for states which do not differ too greatly from the state of thermodynamic equilibrium.* In particular, they are well-adapted to an examination of the *phenomenon of transfer*, since in this case it is solved by the *method of disturbances*, and it is assumed that the velocity distribution does not differ greatly from an equilibrium Maxwell distribution. The phenomena of *electrical conductivity, thermal conductivity, diffusion, and internal friction* in a plasma, for example, refer to the phenomenon of transfer. The *laminar* theory of these phenomena, based on the kinetic Landau equation, has been developed in detail, and comprises a very important part of *plasma kinetics*. It has been described in detail by S. I. Braginskiy (Ref. 19). Therefore, we shall not deal with this theory at all here.

16. DIFFUSION TENSOR AND DYNAMIC FRICTION COEFFICIENT FOR ISOTROPIC DISTRIBUTION OF FIELD PARTICLES IN IMPULSE SPACE.

1. As was noted in Section 14, the vector of particle stream density $\mathbf{I}$ in impulse space - which results from their collisions with each other and with other particles - can be regarded as the overlapping of two streams: the diffusion stream $D_{\alpha\beta}\frac{\partial F}{\partial p_\beta}$ and the stream AF, connected with the dynamic friction coefficient. In terms of its structure, vector $\mathbf{I}$ is completely similar to a stream of regular particles under *forced diffusion* - i.e., diffusion which takes place in the presence of an external force field. Thus, for example, Brownian particles in a viscous liquid, located in a homogeneous field of gravity, are diffused from the location of greatest concentration to the location of least concentration. The forced stream is superimposed on this diffusion stream; the magnitude of

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the forced stream is determined by the combined action of the force of gravity, Archimedes buoyancy force, and the Stokes viscosity force. The density of this forced stream is proportional to the intensity of the field of gravity and the concentration of the particles; in terms of structure, it is completely similar to the term AF.

However, as compared with ordinary diffusion, diffusion of particles (or more accurately, their *representative points*) in impulse space is described by more complex equations. In the first place, the latter equations contain the *diffusion tensor* $D_{\alpha\beta}$, instead of the scalar diffusion coefficient. In the second place, and this is more important, the diffusion tensor $D_{\alpha\beta}$ and the dynamic friction coefficient A do not remain constant, but are determined at each moment of time by the form of the distribution function $F(t, r, p)$ for the plasma particles.

As a result, instead of differential equations, a complex system of *non-linear integro-differential equations* is obtained.

There are no general methods for solving such equations exactly. We must be satisfied with approximate solutions. Moreover, as a rule, the latter can be obtained only after far-reaching schematization and great simplification of the original system of equations. One of the possible simplifications consists of replacing the unknown tensor components $D_{\alpha\beta}$ and the vector A by the known functions of p whose form does not change during the process. The solution which is obtained in this way is called the *diffusion approximation*. The intuitive basis for this can be provided by the fact that - under the usual conditions when the true values of $D_{\alpha\beta}$ and A are replaced by constant quantities which are close to their mean values during the time of the process - it is not possible to change the nature of the diffusion qualitatively, and it is expressed only by the numerical coefficients in the final formulas. It is obvious that the accuracy of the diffusion approximation depends on the extent to which the functions $D_{\alpha\beta}(p)$ and $A_\alpha(p)$ are successfully approximated.

2. Let us assume that the test particle moves in the medium of field particles with isotropic velocity distributions. This means that all of the distribution functions $F^*(p^*)$ of the field particles depend only on the vector length p^*, but not on its direction. Let us find the form of the diffusion tensor $D_{\alpha\beta}$ and the dynamic friction coefficient A in impulse space for such a test particle. Let us direct the z-axis along the impulse p of the test particle, and the x- and y-axes - perpendicular to it. In view of the symmetry in this coordinate system, the tensor $D_{\alpha\beta}$ will have a diagonal form, and in the vector A only the z-component

will differ from zero. Due to the same symmetry, $D_{xx} = D_{yy}$. Let us introduce the designations /151

$$\left.\begin{aligned} D_\perp &= D_{xx} = D_{yy}, \\ D_\parallel &= D_{zz}. \end{aligned}\right\} \tag{16.1}$$

If $\mathbf{I}$ is the stream density vector of test particles in impulse space, then its components along the vector $\mathbf{p}$, which are perpendicular to it, can be written in the form

$$\left.\begin{aligned} \mathbf{I}_\parallel &= -D_\parallel \operatorname{grad}_\parallel F + \mathbf{A}F, \\ \mathbf{I}_\perp &= -D_\perp \operatorname{grad}_\perp F, \end{aligned}\right\} \tag{16.2}$$

where it is obvious that the gradient is chosen in impulse space. The coefficients $D_\parallel$ and $D_\perp$ can be called the *coefficients of longitudinal and transverse diffusion*, respectively.

3. Let us calculate the vector $\mathbf{A}$ and the tensor $D_{\alpha\beta}$ for two cases of isotropic velocity distribution of field particles.

First Case. The velocities $\mathbf{v}^*$ of the same type of field particles are identical in terms of absolute magnitude, but uniformly distributed in terms of direction. For different types of field particles, the magnitudes of these velocities can differ.

In this case, the quantity which is an analog of the electric charge in velocity space is uniformly distributed with respect to a sphere of radius v^*, with the center at the origin. The calculation of the vector $\mathbf{A}$ can be reduced to the elementary problem of electrostatics, concerning the field of a sphere which is uniformly charged with respect to the surface. As can be seen from expression (15.18) and the normalization condition $\int f^*(\mathbf{v}^*)d\mathbf{v}^* = n^*$, the total "charge" distributed on the surface of the sphere equals $4\pi(ee^*)^2 n^*$. Therefore, using the known formulas of electrostatics, we obtain the following from formula (15.15)

$$\mathbf{A} = \sum^* \mathbf{A}(m, m^*), \tag{16.3}$$

where

$$\mathbf{A}(m, m^*) = \begin{cases} -\dfrac{4\pi (ee^*)^2 n^* L}{m^* v^3}\mathbf{v} & \text{for } v > v^*, \\ -\dfrac{2\pi (ee^*)^2 n^* L}{m^* v^3}\mathbf{v} & \text{for } v = v^*, \\ 0 & \text{for } v < v^*. \end{cases} \tag{16.4}$$

As follows from formula (15.16), the calculation of $D_{\alpha\beta}$ can be reduced to calculating the potential function $\Psi(\mathbf{v})$, which is deter-

mined by the expression (15.19). In the case which we are considering of the surface distribution of the "charge", the latter formula changes into

$$\Psi(\mathbf{v}) = \frac{(ee^*)^2 n^*}{2v^{*2}} \int u ds,$$

and integration is carried out with respect to the surface of the sphere under consideration with the radius v^*. This integration can be readily reduced to integration with respect to the angle ϑ (Figure 9): /152

$$\Psi(v) = \pi (ee^*)^2 n^* \int_0^{\pi} u \sin \vartheta d\vartheta.$$

Let us introduce the quantity u as the integration variable. Since $u^2 = v^2 + v^{*2} - 2vv^* \cos \vartheta$, then $udu = vv^* \sin \vartheta d\vartheta$, and after elementary calculations, we obtain

$$\Psi(v) = \begin{cases} 2 (ee^*)^2 n^* \left(v + \frac{1}{3}\frac{v^{*2}}{v}\right) & \text{for } v^* < v, \\ 2\pi (ee^*)^2 n^* \left(v^* + \frac{1}{3}\frac{v^2}{v^*}\right) & \text{for } v^* > v. \end{cases} \tag{16.5}$$

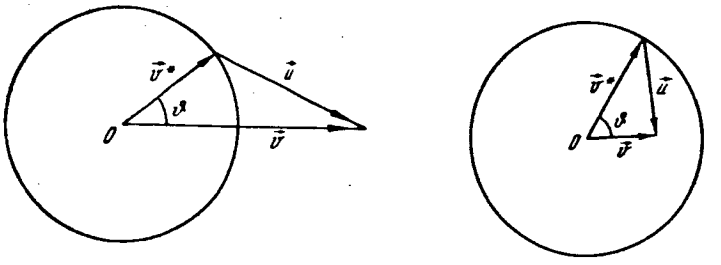

Figure 9.

Now we can readily obtain the following:

$$D_{\alpha\beta} = \sum^* D_{\alpha\beta}(m, m^*), \tag{16.6}$$

where

$$D_{\alpha\beta}(m, m^*) = \begin{cases} \frac{2\pi (ee^*)^2 n^* L}{v} \left[\left(1 - \frac{v^{*2}}{3v^2}\right)\delta_{\alpha\beta} - \right. \\ \left. - \left(1 - \frac{v^{*2}}{v^2}\right)\frac{v_\alpha v_\beta}{v^2}\right] \text{ for } v > v^*, \\ \frac{4\pi (ee^*)^2 n^* L}{3v^*}\delta_{\alpha\beta} \text{ for } v < v^*. \end{cases} \tag{16.7}$$

For $v = v^*$ both expressions coincide.

We can see that, for $v < v^*$, the dynamic friction coefficient

becomes zero, and the diffusion tensor degenerates into a scalar. This result agrees with the statements given in Part 2 of Section 10. The field particles, whose velocity exceeds v, do not influence the longitudinal retardation of the test particle, on the average. Their role only amounts to changing the direction of its motion. This change in the direction of the test particle motion can be statistically described as its diffusion in impulse space with an isotropic diffusion coefficient.

This fact makes it possible to understand the relative role of plasma electrons and ions in the scattering process of a test ion from a new point of view. Let us assume that the velocity of a test ion is on the same order as the mean velocities of thermal motion for plasma ions. With respect to the plasma electrons, we shall assume that their thermal velocities are very large with respect to the ion thermal velocities. Then, very few electrons are found with velocities of $v^* < v$. The effect of these electrons on the scattering process of a test ion can be disregarded. The remaining rapid electrons will cause diffusion of the test ion in impulse space with the diffusion tensor

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$$D_{\alpha\beta}^{ie} = \delta_{\alpha\beta} \cdot \frac{4\pi (ee_e)^2 L}{3} \int \frac{f_e^*(v_e^*)}{v_e^*} dv_e^*.$$

Since the greater velocity v_e^* is in the denominator, the quantity $D_{\alpha\beta}^{ie}$ can be disregarded in comparison with the contribution to the diffusion tensor which the plasma ions introduce. Therefore, in the case under consideration the scattering by electrons can be generally disregarded.

The formulas which have been developed hold for any choice of a coordinate system. In order to obtain the expressions for the longitudinal and transverse diffusion coefficients from them, it is necessary to change to a coordinate system in which one of the coordinate axes is parallel to the vector $\mathbf{v}$. Then the tensor $D_{\alpha\beta}$ becomes diagonal, and we obtain

$$\left.\begin{aligned} D_{\parallel} &= \Sigma^* D_{\parallel}(m, m^*), \\ D_{\perp} &= \Sigma^* D_{\perp}(m, m^*), \end{aligned}\right\} \tag{16.8}$$

where

$$D_{\parallel}(m, m^*) = \begin{cases} \dfrac{4\pi (ee^*)^2 n^* L v^{*2}}{3v^3} & \text{for } v > v^*, \\ \dfrac{4\pi (ee^*)^2 n^* L}{3v^*} & \text{for } v < v^*; \end{cases} \tag{16.9}$$

$$D_{\perp}(m, m^*) = \begin{cases} \dfrac{2\pi (ee^*)^2 n^* L}{v} \left(1 - \dfrac{v^{*2}}{3v^2}\right) & \text{for } v > v^*, \\ \dfrac{4\pi (ee^*)^2 n^* L}{3v^*} & \text{for } v < v^*. \end{cases} \tag{16.10}$$

As follows from the expressions (16.4) and (16.7), the following relationship exists between the diffusion tensor $D_{\alpha\beta}(m, m^*)$ and the dynamic friction coefficient $A_\alpha(m, m^*)$ for $v > v^*$:

$$v_\beta D_{\alpha\beta}(m, m^*) = -\frac{m^* v^{*2}}{3} A_\alpha(m, m^*). \tag{16.11}$$

It gives

$$vD_\parallel(m, m^*) = -\frac{m^* v^{*2}}{3} A(m, m^*). \tag{16.12}$$

For $v < v^*$, these relationships do not hold, since in this case $A_\alpha(m, m^*) = 0$, while $D_{\alpha\beta} \neq 0$.

Second Case. The velocities of the field ions and electrons have a Maxwell distribution. The ion and electron temperatures cannot coincide. In this case, A and $D_{\alpha\beta}$ are represented by the expressions (16.3) and (16.6), following the preceding method, while we obtain the following from formulas (15.15) and (6.3): /154

$$\mathbf{A}(m, m^*) = -\frac{4\pi (ee^*)^2 n^* L}{m^* v^3} \Phi_1(b^* v)\,\mathbf{v}. \tag{16.13}$$

We can determine the potential function $\Psi(v)$ by integration of the function (16.5) with respect to the Maxwell distribution

$$f^*(v^*) = n^* \left(\frac{b^*}{\sqrt{\pi}}\right)^3 e^{-b^{*2}v^{*2}}.$$

This gives

$$\Psi(v) = 2\pi (ee^*)^2 n^* \left\{ \int_0^v \left(v + \frac{1}{3}\frac{v^{*2}}{v}\right) f^*(v^*) \cdot 4\pi v^{*2} dv^* + \right.$$
$$\left. + \int_v^\infty \left(v^* + \frac{1}{3}\frac{v^2}{v^*}\right) f^*(v^*) \cdot 4\pi v^{*2} dv^* \right\}.$$

By carrying out the integration, we obtain

$$\Psi(v) = 2\pi (ee^*)^2 n^* v \left[\left(1 + \frac{1}{2b^{*2}v^2}\right) \Phi(b^* v) + \frac{1}{\sqrt{\pi} b^* v} e^{-b^{*2}v^2} \right]. \tag{16.14}$$

After this, with the aid of formula (15.16) we find

$$D_{\alpha\beta}(m, m^*) = \frac{2\pi (ee^*)^2 n^* L}{v} \left[\left(\Phi(b^* v) - \frac{\Phi_1(b^* v)}{2b^{*2} v^2}\right) \delta_{\alpha\beta} - \right.$$
$$\left. - \left(\Phi(b^* v) - \frac{3\Phi_1(b^* v)}{2b^{*2} v^2}\right) \frac{v_\alpha v_\beta}{v^2} \right], \tag{16.15}$$

$$D_\parallel(m, m^*) = \frac{2\pi (ee^*)^2 n^* L}{v} \frac{\Phi_1(b^* v)}{b^{*2} v^2}, \tag{16.16}$$

$$D_\perp(m, m^*) = \frac{2\pi (ee^*)^2 n^* L}{v} \left[\Phi(b^* v) - \frac{\Phi_1(b^* v)}{2b^{*2} v^2} \right]. \tag{16.17}$$

From formulas (16.13) and (16.15), we have

$$v_\beta D_{\alpha\beta}(m, m^*) = -\frac{m^*}{2b^{*2}} A_\alpha(m, m^*)$$

or

$$v_\beta D_{\alpha\beta}(m, m^*) = -T^* A_\alpha(m, m^*). \tag{16.18}$$

In particular we have

$$vD_\parallel(m, m^*) = -T^* A(m, m^*). \tag{16.19}$$

The density of the particle stream I_α in impulse space, which is caused by collisions, is determined by expression (14.10). If the velocities of all the types of particles have a Maxwell distribution, but the temperatures of different types of particles can differ from each other, then we can write the following in view of the relationship (16.18): /155

$$I_\alpha = \frac{F}{T} \sum^* A_\alpha(m, m^*)\,[T - T^*]. \tag{16.20}$$

In the state of thermodynamic equilibrium, when the temperatures of all the plasma components are identical, this expression becomes zero. This means that the collisions between particles does not affect the distribution function. It could be shown that this condition is a necessary condition for complete thermodynamic equilibrium, and on this basis it could also be shown that the only distribution which satisfies this condition is the Maxwell velocity distribution.

4. The form of formulas (16.13), (16.16) and (16.17) is simplified when a rapid electron is the test particle - i.e., one whose velocity v is large as compared with the mean velocity of thermal motion of electrons. For purposes of simplicity, we shall assume that the plasma consists of electrons and one type of positively-charged ions. Then, in the calculation of $\mathbf{A}$ and $D_\parallel$ the influence of the ions due to their relatively large mass can be disregarded. For this reason, in a calculation of $D_\perp$ it is possible to disregard the term $\dfrac{\Phi_1(b^*v)}{2b^{*2}v^2}$, when the collisions of a test electron with ions are examined. This term must be retained only for collisions of a test electron with electrons. Thus, since the argument b^*v is always large (it is sufficient for it to exceed 2), then the functions $\Phi(b^*v)$ and $\Phi_1(b^*v)$ can be approximated by unity. As a result, we obtain

$$\mathbf{A} = -\nu(p)\,\mathbf{p}, \tag{16.21}$$

$$D_\parallel = \nu(p)\,mT_e, \tag{16.22}$$

$$D_\perp = \nu(p)\,p^2\left(1 - \frac{mT_e}{2p^2}\right), \tag{16.23}$$

where m is the electron mass. For purposes of brevity, the following quantity is designated by $\nu(p)$:

$$\nu(p) = \frac{4\pi e^4 nL}{m^2 v^3} \doteq \frac{4\pi e^4 mnL}{p^3}, \tag{16.24}$$

which signifies the frequency of collisions between a test electron and electrons or ions in the plasma (it is assumed that the plasma is quasi neutral).

The expressions obtained can be used to describe the kinetic equation for *rapid* electrons. For example, in an investigation of the phenomenon of electron escape in weak electric fields (i.e., fields which are much less than the critical field) all of the electrons can be divided into two groups: electrons of the main group - whose distribution function is almost a Maxwell distribution - and electrons which have high velocities. If the concentration of the latter electrons is small as compared with the concentrations of the electrons in the primary group, it is possible to disregard the scattering by rapid electrons. It is sufficient to consider the scattering of rapid electrons by electrons in the primary group and by ions. Therefore, in a description of the kinetic equation for rapid electrons, it is possible to use expressions (16.21), (16.22) and (16.23). /156

According to equation (14.10), we have

$$\mathbf{I} = \mathbf{A}F - D_{\parallel} \operatorname{grad}_{\parallel} F - D_{\perp} \operatorname{grad}_{\perp} F, \tag{16.25}$$

where the gradient is chosen in impulse space. By way of an example, let us conceive of a spherical coordinate system in this space with an axis which is directed along the effective electric force $e\mathbf{E}$. It is assumed that the electric field is uniform. The distribution function F, in addition to time, depends on p and on the polar angle ϑ. It does not depend on the azimuth φ, due to the assumption of cylindrical symmetry for the problem. Therefore, vector $\mathbf{I}$ has only two components: the component I_p along the vector $\mathbf{p}$, and the component I_ϑ which is perpendicular to it, and whose positive direction coincides with the direction in which the angle ϑ increases. For these components, we have

$$\left.\begin{aligned} I_p &= AF - D_{\parallel} \frac{\partial F}{\partial p}, \\ I_\vartheta &= -D_{\perp} \frac{\partial F}{p \partial \vartheta}, \end{aligned}\right\} \tag{16.26}$$

or

$$\left.\begin{aligned} I_p &= -\nu(p) \left[mT_e \frac{\partial F}{\partial p} + pF \right], \\ I_\vartheta &= -\nu(p)\, p \left(1 - \frac{mT_e}{2p^2}\right) \frac{\partial F}{\partial \vartheta}. \end{aligned}\right\} \tag{16.27}$$

Utilizing these formulas, and also the expression for divergence $\frac{\partial I_\alpha}{\partial p_\alpha}$ in a spherical coordinate system, we can attribute the following form to equation (14.9):

$$\frac{\partial F}{\partial t} + (\mathbf{v}\nabla) F + eE\left(\cos\vartheta \frac{\partial F}{\partial p} - \frac{\sin\vartheta}{p}\frac{\partial F}{\partial \vartheta}\right) -$$

$$- \frac{1}{p^2}\frac{\partial}{\partial p}\left[\nu p^2\left(mT_e\frac{\partial F}{\partial p} + pF\right)\right] - \frac{\nu}{\sin\vartheta}\left(1 - \frac{mT_e}{2p^2}\right)\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial F}{\partial\vartheta}\right) = 0.$$

A. V. Gurevich used this equation in his work on the theory of electron escape (Ref. 16) [1].

17. APPLICATION OF THE KINETIC EQUATION TO THE PROBLEM OF ENERGY EXCHANGE BETWEEN DIFFERENT PLASMA COMPONENTS

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1. The problem of energy exchange between different plasma components was solved in Section 9. Let us introduce another solution for this problem, based on a kinetic equation. We shall assume that there are no external force fields.

In the derivation of all the primary equations, it was assumed that the total energy does not change in the collisions between any two particles. *The collisions can not change the total energy supply of plasma particles, but only redistribute it between these particles.* It can be readily verified that the expression for I_α satisfies this condition. We shall use $\mathcal{E}$ to designate the energy of any type of plasma particle per unit of volume

$$\mathcal{E} = \int \frac{p^2}{2m} F(t, \mathbf{p})\, d\mathbf{p}. \qquad (17.1)$$

For its time derivative, we can write

$$\frac{\partial \mathcal{E}}{\partial t} = \int \frac{p^2}{2m}\frac{\partial F}{\partial t}\, d\mathbf{p}.$$

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[1] In the work of A. V. Gurevich, the last term in this equation differs somewhat from our term: instead of the correct factor $(1 - \frac{mT_e}{2p^2})$, $(1 - \frac{mT_e}{4p^2})$ erroneously appears in his work.

In the case which we are considering, $X_\alpha = 0$. In addition, it is assumed that the plasma is uniform, and consequently $\frac{\partial F}{\partial x_\alpha} = 0$. Therefore, equation (14.9) assumes the form

$$\frac{\partial F}{\partial t} + \frac{\partial I_\alpha}{\partial p_\alpha} = 0, \tag{17.2}$$

and we obtain the following:

$$\frac{\partial \mathcal{E}}{\partial t} = -\int \frac{p^2}{2m} \frac{\partial I_\alpha}{\partial p_\alpha} d\mathbf{p}.$$

Let us transform the right side according to the Gauss theorem into the following form:

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{1}{m} \int I_\alpha p_\alpha \, d\mathbf{p}. \tag{17.3}$$

This expression can be summed over all the particle types, and it is found that zero is obtained as a result. For this purpose, the simplest method is to use the expression for I_α in symmetrical Landau form (15.10). We then obtain

$$\frac{\partial}{\partial t} \sum \mathcal{E} = \sum\sum^* 2\tau (ee^*)^2 L \iint F(\mathbf{p}) \frac{\partial F^*(\mathbf{p}^*)}{\partial p^*_\beta} v_\alpha u_{\alpha\beta} d\mathbf{p}\, d\mathbf{p}^* -$$
$$- \sum\sum^* 2\tau (ee^*)^2 L \iint F(\mathbf{p}^*) \frac{\partial F(\mathbf{p})}{\partial p_\beta} v_\alpha u_{\alpha\beta} d\mathbf{p}\, d\mathbf{p}^*.$$

The symbol Σ^* is used for summation of the quantities designated by the letters with asterisks, and the symbol Σ - for summation of the quantities designated by letters without an asterisk. Since we are discussing double summation over one and the same particle types, all of the quantities with an asterisk can be replaced by the corresponding quantities without an asterisk, and vice versa. Let us make this substitution in the second component of the preceding expression. Then, taking the fact into consideration that $u^*_{\alpha\beta} = u_{\alpha\beta}$, $v_\alpha - v^*_\alpha = u_\alpha$, we obtain

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$$\frac{\partial}{\partial t} \sum \mathcal{E} = \sum\sum^* 2\pi (ee^*)^2 L \iint F(\mathbf{p}) \frac{\partial F^*(\mathbf{p}^*)}{\partial p^*_\beta} u_\alpha u_{\alpha\beta} d\mathbf{p}\, d\mathbf{p}^*.$$

As can be readily verified with the aid of formula (15.4):

$$u_\alpha u_{\alpha\beta} = 0 \tag{17.4}$$

and, consequently,

$$\frac{\partial}{\partial t} \sum \mathcal{E} = 0. \tag{17.5}$$

2. Let us make a more detailed study of the energy change $\mathcal{E}$ in the group of particles under consideration due to their collisions with other plasma particles. We shall assume that the velocity of all the groups has a Maxwell distribution with different temperatures. Substituting expression (16.20), in formula (17.3), we obtain

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{1}{T} \sum\nolimits^{*} (T - T^*) \int v_\alpha A_\alpha (m, m^*) F(\mathbf{p})\, d\mathbf{p}. \qquad (17.6)$$

Just as in Section 9, let us focus our attention on only one component of the sum (17.6) - i.e., we shall examine the energy exchange only between two particle groups. For this component, we can write

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{T - T^*}{T} \int vA(m, m^*) F(\mathbf{p})\, d\mathbf{p} = \frac{T - T^*}{T} \int vA(m, m^*) f(\mathbf{v})\, d\mathbf{v}$$

or

$$\frac{\partial \mathcal{E}}{\partial t} = n \frac{T - T^*}{T} \langle vA(m, m^*) \rangle, \qquad (17.7)$$

where the averaging is carried out with respect to the Maxwell distribution of test particles (particles of the first group). Instead of $A(m, m^*)$, substituting its expression from formula (16.13), we obtain

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{T^* - T}{T} \cdot \frac{4\pi (ee^*)^2 n n^* L}{m^*} \left\langle \frac{\Phi_1 (b^* v)}{v} \right\rangle.$$

In calculating $\left\langle \frac{\Phi_1 (b^* v)}{v} \right\rangle$, we can use the results obtained at the beginning of Section 9. In view of formula (6.6), they give

$$\left\langle \frac{\Phi_1 (b^* v)}{v} \right\rangle = \frac{2}{\sqrt{\pi}} \frac{bb^{*3}}{(b^2 + b^{*2})^{3/2}}.$$

Finally, we have

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$$\frac{\partial \mathcal{E}}{\partial t} = \frac{T^* - T}{mm^*} \frac{4\sqrt{2\pi} n n^* (ee^*)^2 L}{\left(\frac{T}{m} + \frac{T^*}{m^*}\right)^{3/2}}. \qquad (17.8)$$

This result agrees with formula (9.2), since

$$Q = -\frac{1}{n} \frac{\partial \mathcal{E}}{\partial t}.$$

18. THE OUTFLOW OF IONS FROM A MAGNETIC TRAP WITH MAGNETIC MIRRORS AS A RESULT OF COLLISIONS

1. Let us apply a kinetic equation to the problem of outflow

of ions from a *magnetic trap with magnetic mirrors*, which was proposed by G. I. Budker (Ref. 20), and independently of him by York (Ref. 21) for the *confinement* of charged particles. The trap represents a cylindrical tube placed in a solenoid, which creates a strong, uniform, constant magnetic field B, which is parallel to the axis of the trap. At the ends of the trap there are auxiliary windings which intensify the magnetic field. The regions of the intensified, constant magnetic field at the ends of the trap are called *the magnetic mirrors*. We shall use B to designate the intensity of a uniform magnetic field in the trap, and B_{max} to designate the maximum value of the latter in the magnetic mirrors (it is assumed that B_{max} is the same at both ends of the trap). In addition, we shall use ϑ to designate the angle between the direction of motion for a particle in the trap and the direction of the magnetic field in those regions where it can be assumed to be uniform. As is known, the *drift theory* for the motion of a charged particle in magnetic fields leads to the following results. *A charged particle is confined in the trap for an infinite period of time, if* $\pi - \vartheta_0 > \vartheta > \vartheta_0$, *and it leaves the trap through the mirror if* $\vartheta < \vartheta_0$ *or* $\vartheta > \pi - \vartheta_0$. Here ϑ_0 *is the limiting angle*, which is determined by the relationship

$$\sin \vartheta_0 = \sqrt{\frac{B}{B_{max}}}. \qquad (18.1)$$

If there is not one particle in the trap, but only the plasma, then in this case even in the drift approximation there is an *escape of particles through the mirror*. One of the reasons for this escape is provided by the *Coulomb collisions*, which are accompanied by changes in the angle ϑ and changes in the points representing the outgoing particles (in impulse space or velocity space) within the *limiting cone* $\vartheta < \vartheta_0$, $\vartheta > \pi - \vartheta_0$. These collisions lead to the outflow of particles from the trap through the magnetic mirrors. We shall only study here the mechanism for the departure of the particles, and we shall not concern ourselves with other mechanisms - for example, *collective fluctuations and instabilities of the plasma* [1]. /160

[1] It is known that convective plasma instability occurs due to a decrease in the intensity of the magnetic field from the center toward the periphery in the magnetic trap, which was proposed by G. I. Budker. Therefore, it would appear that we are not interested in the problem of the outflow of particles through the magnetic mirrors due to collisions, within the framework in which it is being examined here. In fact, this is not absolutely the case. By the superposition of the supplementary magnetic fields, which provide for an increase in the

2. The solution of this problem can be reduced to determining the distribution function $F(t, \mathbf{r}, p)$ for the type of particles under consideration. If new particles do not enter the trap, this problem is *non-stationary*. In order to simplify the solution, following the procedure of G. I. Budker (Ref. 20) we shall turn to the stationary problem. For this purpose, we shall introduce the *sources* of the new particles, which balance the outflow of particles from the trap. Idealizing the problem, we shall assume that these sources are continuously distributed throughout the entire trap. Then, instead of equation (14.4), we must write

$$\frac{\partial F}{\partial t} + \frac{\partial I_{\alpha}^{(6)}}{\partial \xi_{\alpha}} = q\left(\vec{\xi}\right), \tag{18.2}$$

where $q\left(\vec{\xi}\right)$ is the *strength density of the sources in phase space*, i.e., the number of particles produced by the sources per unit of volume of phase space in one second. In three-dimensional form, equation (18.2) is

$$\frac{\partial F}{\partial t} + v_{\alpha}\frac{\partial F}{\partial x_{\alpha}} + X_{\alpha}\frac{\partial F}{\partial p_{\alpha}} + \frac{\partial I_{\alpha}}{\partial p_{\alpha}} = q(\mathbf{r}, \mathbf{p}). \tag{18.3}$$

The stream $\mathbf{I}$ arises as a result of the collisions between the particles under consideration, and collisions between these and other particles. Therefore, its expression does not explicitly depend on the form of the function $q(\mathbf{r}, \mathbf{p})$ but is entirely determined by the distribution functions of these plasma particles at the moment of time under consideration. This means that in the presence of the sources formula (15.10) is valid, as well as all of the expressions which are equivalent to it.

In order to simplify the problem further, we shall assume that, in the first place, the plasma is a two-component plasma - i.e., it consists of electrons and the same type of ions. It is assumed

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(Footnote continued from previous page)

intensity of the magnetic field toward the periphery, it is possible to suppress the convective instability. However, in impulse space all of the "gaps" - which are similar to the limiting cone in the Budker trap - are closed, through which the particles must pass from the trap as a result of the collisions. The problem regarding the departure of particles through such gaps is fully analogous to the problem which we are discussing regarding the departure of particles through the limiting cone in the Budker trap. Therefore, the results which we obtained will be applied both qualitatively, and to a significant extent quantitatively, to traps containing gaps in which there is no convective instability.

that the electron velocities are very large as compared with the ion velocities; it is then possible to disregard the scattering of the ions by the electrons. In the second place, we shall examine a simplified model of the trap, assuming that throughout almost all of the trap the magnetic field B is uniform, is parallel to its axis, and very rapidly (within a limit - discontinuously) increases at the edge of the trap up to the maximum value B_{max}. With this simplification, the magnetic field in the mirrors will not enter into equation (18.3), but only enters into its boundary conditions. /161

In the stationary state, the function F can depend only on the length of the vector p and on the angle ϑ, which it forms with the axis of the trap. The vector I will lie in the *meridian plane* and, in conformance with this, will have only two components: *longitudinal* component I_p along the vector p, and the *meridian* component I_ϑ, which is perpendicular to it. There will be no *azimuth* component I_φ. Therefore, as can be readily seen, equation (18.3) can be written in the form

$$p\frac{\partial}{\partial\vartheta}(\sin\vartheta I_\vartheta)+\sin\vartheta\frac{\partial}{\partial p}(p^2I_p)=qp^2\sin\vartheta. \qquad (18.4)$$

The terms $\frac{\partial F}{\partial t}$ and $v_\alpha\frac{\partial F}{\partial x_\alpha}$ become zero. The term $X_\alpha\frac{\partial F}{\partial p_\alpha}$ in the case under consideration is reduced to $\frac{e}{c}[\mathrm{vB}]\frac{\partial F}{\partial \mathrm{p}}$ and also becomes zero, since the vectors v, B, $\frac{\partial F}{\partial \mathrm{p}}$ are coplanar. We obviously shall assume that condition (15.21) is fulfilled.

On the basis of formula (14.10), for I_ϑ and I_p we can write

$$\left.\begin{aligned} I_\vartheta &= -\frac{1}{p}D_{\vartheta\vartheta}\frac{\partial F}{\partial\vartheta}-D_{\vartheta p}\frac{\partial F}{\partial p}+A_\vartheta F,\\ I_p &= -D_{pp}\frac{\partial F}{\partial p}-\frac{1}{p}D_{p\vartheta}\frac{\partial F}{\partial\vartheta}+A_pF, \end{aligned}\right\} \qquad (18.5)$$

while in view of formula (14.12), we have:

$$D_{\vartheta p}=D_{p\vartheta}.$$

Boundary conditions must be added to equation (18.4). Since the time in which a particle is confined in the trap is very large as compared with the time of its free flight from one end of the trap to another, we can assume - just as G. I. Budker did - that

within the limiting cone $\vartheta < \vartheta_0$, $\vartheta > \pi - \vartheta_0$ there are absolutely no particles. Therefore, we require that the function F becomes zero both within and at the boundaries of the given cone. However, equation (18.4) does not change if angle ϑ is replaced by angle $\pi - \vartheta$. It thus follows from the boundary conditions that $F(\vartheta) = F(\pi - \vartheta)$. In addition, this is apparent from the symmetry of the problem. Therefore, in solving equation (18.4), we can confine ourselves to the integral of the angles $\vartheta_0 \leqslant \vartheta \leqslant \frac{\pi}{2}$, and can write the boundary conditions in the following form:

$$\left.\begin{aligned} F &= 0 \text{ for } \vartheta = \vartheta_0, \\ \frac{\partial F}{\partial \vartheta} &= 0 \text{ for } \vartheta = \frac{\pi}{2}. \end{aligned}\right\} \tag{18.6}$$

3. An exact solution of this problem is difficult, because the tensor $D_{\alpha\beta}$ and the vector A are not known, and are determined by the mode of the unknown distribution function F. In order to make a solution possible, let us change to the *diffusion approximation*, in which $D_{\alpha\beta}$ and A are replaced by the known functions of p and ϑ. Unfortunately, it is impossible to manage without introducing unjustified and arbitrary assumptions, whose effect on the nature of the solution is difficult to estimate. However, it can be assumed that in a successful substitution the basic features of the phenomenon which interests us will not be touched upon. Let us first replace $D_{\alpha\beta}$ and A_α by the values which they would assume with an isotropic distribution function F, without specifying the form of this function. Then $D_{\vartheta p}$ and A_ϑ become zero, and the components $D_{pp} \equiv D_\parallel$, $D_{\vartheta\vartheta} \equiv D_\perp$ and $A_p \equiv A$ become functions only of p, and will not depend on ϑ. Therefore, equation (18.4) assumes the form

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$$D_\perp \frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial F}{\partial \vartheta}\right) + \sin\vartheta \frac{\partial}{\partial p}\left[p^2\left(D_\parallel \frac{\partial F}{\partial p} - AF\right)\right] = \\ = -qp^2 \sin\vartheta. \tag{18.7}$$

Let us now conceive of the expressions for $D_\parallel$, $D_\perp$ and A with a *Maxwell* velocity distribution, and we shall search for the function F in the form

$$F = \Theta(\vartheta)\, e^{-\frac{p^2}{2mT}}. \tag{18.8}$$

Then, in view of the relationship (16.19), we have

$$D_\parallel \frac{\partial F}{\partial p} - AF = -\left[\frac{v}{T} D_\parallel + A\right] F = 0$$

and equation (18.7) changes to

$$\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial F}{\partial\vartheta}\right)=-\frac{qp^2}{D_\perp}\sin\vartheta. \tag{18.9}$$

4. We assume that the function q depends only on p, but does not depend on ϑ (*isotropic injection*). Then the solution of the equation (18.9), which satisfies the boundary conditions (18.6), assumes the form

$$F=\frac{qp^2}{D_\perp}\ln\frac{\sin\vartheta}{\sin\vartheta_0}. \tag{18.10}$$

Comparing this expression with expression (18.8), we conclude that

$$\Theta(\vartheta)=C\ln\frac{\sin\vartheta}{\sin\vartheta_0}, \tag{18.11}$$

$$\frac{qp^2}{D_{\perp(p)}}=Ce^{-\frac{p^2}{2mT}}, \tag{18.12}$$

where C is the constant. Formula (18.12) determines the strength density $q(p)$ of the forces, at which a *quasi-Maxwell* distribution of the type (18.8) is maintained in the plasma. As can be seen from formula (16.17), relationship $\frac{D_\perp(p)}{p^2}$ decreases with an increase in p : for small values of p, it is proportional to $\frac{1}{p^2}$; for large values - it is proportional to $-\frac{1}{p^3}$. Therefore, in comparison with the Maxwell distribution, the sources must have relatively more slow ions than they do rapid ions. This is understandable, since the slow ions diffuse more rapidly within the limiting cone $\vartheta<\vartheta_0, \vartheta>\pi-\vartheta_0$, than do the rapid ions. /163

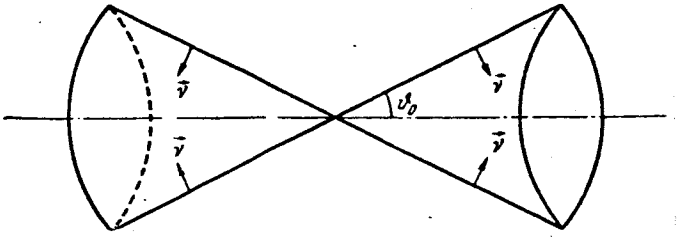

Figure 10.

Thus, we have:

$$F=C\ln\frac{\sin\vartheta}{\sin\vartheta_0}e^{-\frac{p^2}{2mT}}, \tag{18.13}$$

while the constant C is determined from the normalization condition

$$\int F(\vartheta, p)\, d\mathbf{p} = n. \tag{18.14}$$

5. By knowing the distribution function $F(\vartheta, p)$, we can determine the *rate at which the ions leave* the trap as a result of the collisions. It is determined by the number N of ions which leave per unit of volume of the trap in one second through the surface of the limiting cone $\vartheta = \vartheta_0$, $\vartheta = \pi - \vartheta_0$. It apparently equals

$$N = \int I_\nu\, dS,$$

where integration is carried out with respect to the surface of the entire limiting cone, and $\vec{\nu}$ is the unit vector of the internal normal to this surface - i.e., of the external one with respect to the impulse space occupied by the particles (Figure 10). By partly closing this surface of the infinite sphere, which passes outside of the limiting cone, transforming the integral by the Gauss theorem, and going to the limit , we can write

$$N = \int \frac{\partial I_\alpha}{\partial p_\alpha}\, d\mathbf{p},$$

while integration is carried out with respect to the entire impulse space outside of the limiting cone. In view of equation (18.3), we have $\frac{\partial I_\alpha}{\partial p_\alpha} = q$, and therefore

$$N = \int\limits_{\vartheta_0 < \vartheta \leq \pi - \vartheta_0} q\, d\mathbf{p}. \tag{18.15}$$

This must be the case, since in a stationary state the number of ions which are supplied by the source must equal the number of outgoing ions. /164

6. The problem is reduced to calculating two integrals: *the normalization integral* (18.14) and the integral (18.15). It is advantageous to take the following as the volume element of impulse space:

$$d\mathbf{p} = 2\pi p^2 \sin\vartheta\, d\vartheta\, dp. \tag{18.16}$$

Then, after elementary calculations, we can readily obtain

$$C \cdot (2\pi m T)^{3/2} \left[\ln \operatorname{ctg} \frac{\vartheta_0}{2} - \cos\vartheta_0\right] = n, \tag{18.17}$$

$$N = 4\pi C \cos\vartheta_0 \int\limits_0^\infty D_\perp(p)\, e^{-\frac{p^2}{2mT}} dp. \tag{18.18}$$

According to formula (16.17):

$$D_{\perp} = \frac{2\pi e^4 nmL}{\sqrt{2mT}x}\left[\Phi(x) - \frac{\Phi_1(x)}{2x^2}\right], \qquad (18.19)$$

where $x = v\sqrt{\frac{m}{2T}} = \frac{p}{\sqrt{2mT}}$. The substitution of this expression in the preceding formula gives

$$N = 8\pi^2 e^4 nmLC \int_0^{\infty} \frac{1}{x}\left[\Phi(x) - \frac{\Phi_1(x)}{2x^2}\right] e^{-x^2} dx. \qquad (18.20)$$

In order to calculate the integrals entering into this, we shall first determine the integral

$$I(b) = \int_0^{\infty} \frac{\Phi(x)}{x} e^{-bx^2} dx.$$

The integral, which is obtained by differentiation of the integrand with respect to the parameter b, converges uniformly with respect to b in any interval $\alpha \leqslant x < +\infty$, where $\alpha > 0$. Therefore, we have

$$\frac{dI}{db} = -\int_0^{\infty} x\Phi(x) e^{-bx^2} dx = -\frac{2}{\sqrt{\pi}} \int_0^{\infty} x e^{-bx^2} dx \int_0^{x} e^{-y^2} dy.$$

Changing the order of integration, we find

$$\frac{dI}{db} = -\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-y^2} dy \int_y^{\infty} x e^{-bx^2} dx = -\frac{1}{2b\sqrt{1+b}}.$$

Since $I(\infty) = 0$, then

$$I(b) = -\int_{\infty}^{b} \frac{dx}{2x\sqrt{1+x}} = \ln \frac{\sqrt{1+b}+1}{\sqrt{b}}.$$

Consequently, /165

$$\int_0^{\infty} \frac{\Phi(x)}{x} e^{-x^2} dx = I(1) = \ln(\sqrt{2}+1).$$

In order to calculate the second integral in formula (18.20) we should note that

$$\Phi_1(x) = \Phi(x) - x\Phi'(x),$$
$$\Phi'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2},$$

and therefore we can write

$$\int_0^{\infty} \frac{\Phi_1(x)}{2x^2} e^{-x^2} dx = -\frac{\sqrt{\pi}}{8} \int_0^{\infty} [\Phi(x) - x\Phi'(x)]\Phi'(x)\, d\frac{1}{x^2}.$$

Thus, performing integration by parts, we find

$$\int_0^\infty \frac{\Phi_1(x)}{2x^3} e^{-x^2} dx = \frac{\sqrt{\pi}}{8} \int_0^\infty \frac{\Phi\Phi'' - 2x\Phi'\Phi''}{x^2} dx =$$

$$= -\frac{1}{2} \int_0^\infty \frac{\Phi(x)}{x} e^{-x^2} dx + \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-2x^2} dx =$$

$$= -\frac{1}{2} \ln(\sqrt{2}+1) + \frac{1}{\sqrt{2}}.$$

Consequently,

$$N = 4\pi^2 e^4 nmLC \left[3 \ln(\sqrt{2}+1) - \sqrt{2}\right] \cos\vartheta_0. \qquad (18.21)$$

Eliminating the constant C both here and from formula (18.17), we obtain

$$N = \sqrt{\frac{2\pi}{m}} \cdot \frac{Le^4 n^2}{T^{3/2}} \cdot \frac{3 \ln(\sqrt{2}+1) - \sqrt{2}}{\ln \operatorname{ctg} \frac{\vartheta_0}{2} - \cos\vartheta_0} \cos\vartheta_0. \qquad (18.22)$$

This formula was first obtained by G. I. Budker (Ref. 20) by a similar method, which differs from that employed here only in terms of the calculations.

7. The assumption which was made in deriving formula (18.22)- to the effect that the function q is not dependent on the angle ϑ (isotropic injection) - is of little importance. In order to confirm this fact, let us assume that q has the form

$$q = q_0(p)\,\varphi(\vartheta), \qquad (18.23)$$

where $\varphi(\vartheta)$ is arbitrary within wide limits of the function. Let us normalize it, so that

$$\int_{\vartheta_*}^{\frac{\pi}{2}} \varphi(\vartheta) \sin\vartheta\, d\vartheta = \cos\vartheta_0, \qquad (18.24)$$

- i.e., so that this integral assumes the same value as for isotropic injection, when $\varphi(\vartheta) = 1$. /166

The solution of equation (18.9), which satisfies the boundary conditions (18.6), will be

$$F = \frac{q_0 p^2}{D_\perp} \int_{\vartheta_*}^{\vartheta} \frac{d\vartheta'}{\sin\vartheta'} \int_{\vartheta'}^{\frac{\pi}{2}} \varphi(\vartheta'') \sin\vartheta''\, d\vartheta''. \qquad (18.25)$$

With the aid of the well-known *mean-value theorem* of calculus, this expression can be written in the form

$$F = \frac{q_0 p^2}{D_\perp} \varphi(\vartheta_1) \ln \frac{\sin \vartheta}{\sin \vartheta_0}, \tag{18.26}$$

where ϑ_1, generally speaking, depends on ϑ and satisfies the condition

$$\vartheta_0 \leqslant \vartheta_1 \leqslant \frac{\pi}{2}. \tag{18.27}$$

Now, instead of formulas (18.11) and (18.12) we must write

$$\Theta(\vartheta) = C\varphi(\vartheta_1) \ln \frac{\sin \vartheta}{\sin \vartheta_0}, \tag{18.28}$$

$$\frac{q_0 p^2}{D_\perp(p)} = Ce^{-\frac{p^2}{2mT}}. \tag{18.29}$$

Expression (18.21) for N does not change, in view of the normalization condition (18.24). The entire difference is reduced to calculating the normalization integral (18.14). Instead of the previous integral,

$$\int_{\vartheta_*}^{\frac{\pi}{2}} \sin \vartheta \ln \frac{\sin \vartheta}{\sin \vartheta_0} d\vartheta$$

we must calculate the integral

$$\int_{\vartheta_*}^{\frac{\pi}{2}} \varphi(\vartheta_1) \sin \vartheta \ln \frac{\sin \vartheta}{\sin \vartheta_0} d\vartheta.$$

But, according to the mean-value theorem this integral must be represented in the form

$$\bar{\varphi} \int_{\vartheta_*}^{\frac{\pi}{2}} \sin \vartheta \ln \frac{\sin \vartheta}{\sin \vartheta_0} d\vartheta,$$

where $\bar{\varphi}$ is the value of the function $\varphi(\vartheta)$ for any intermediate value of the argument between ϑ_0 and $\frac{\pi}{2}$. This value appears as a multiplier in the left part of the relationship (18.17). Therefore, instead of formula (18.22), the following formula is obtained which barely differs from it: /167

$$N = \sqrt{\frac{2\pi}{m}} \frac{Le^4 n^2}{T^{3/2}} \cdot \frac{3 \ln(\sqrt{2}+1) - \sqrt{2}}{\bar{\varphi}\left[\ln \operatorname{ctg} \frac{\vartheta_0}{2} - \cos \vartheta_0\right]} \cos \vartheta_0. \tag{18.30}$$

8. G. I. Budker (Ref. 20) also examines the case of particle

injection perpendicular to the axis of the trap. In this case, the function $\varphi(\vartheta)$ has the form

$$\varphi(\vartheta) = \cos\vartheta_0 \delta\left(\vartheta - \frac{\pi}{2}\right), \tag{18.31}$$

and formula (18.25) gives

$$F = \frac{q_0\rho^2}{D_\perp}\cos\vartheta_0 \ln\frac{\operatorname{tg}\frac{\vartheta}{2}}{\operatorname{tg}\frac{\vartheta_0}{2}}. \tag{18.32}$$

With the aid of this formula, we find

$$N = \sqrt{\frac{2\pi}{m}}\frac{Le^4n^2}{T^{3/2}}\cdot\frac{3\ln(\sqrt{2}+1)-\sqrt{2}}{\ln\frac{1}{\sin\vartheta_0}}\cos\vartheta_0. \tag{18.33}$$

As follows from the inequality which can be readily proved

$$\ln\operatorname{ctg}\frac{\vartheta_0}{2} - \cos\vartheta_0 < \ln\frac{1}{\sin\vartheta_0},$$

for one and the same values of the parameters n, T and ϑ_0, formula (18.33) yields smaller values for N than does formula (18.22). This is understandable, since in the case of particle injection perpendicular to the magnetic field they must change the direction of their motion by an angle, which is not less than $\frac{\pi}{2} - \vartheta_0$, in order to leave the trap.

9. The time τ_{out*}, during which n particles leave a unit of volume in the trap is determined by the expression

$$\tau_{out} = \frac{n}{N}. \tag{18.34}$$

In a stationary state, the source injects the same number of new particles into the trap at the location of the outgoing particles. If there were no source, then - during the time determined by formula (18.34) - not all of the particles would be able to leave the trap. However, in this case the time τ_{out} would determine in order of magnitude that time interval in which the main portion of the particles leaves the trap. Therefore, we can designate the quantity τ_{out} as the *mean confinement time of a particle in the trap*.

For isotropic injection, the time τ_{out} can be found if expression (18.22) is substituted in formula (18.34), instead of N. Thus combining all of the numerical multipliers, we arrive at the following formula: /168

*Note: out designates 'outgoing'.

$$\tau_{out} = 1{,}57 \cdot \frac{\ln \operatorname{ctg} \frac{\vartheta_0}{2} - \cos \vartheta_0}{\cos \vartheta_0} \bar{\tau}_i, \qquad (18.35)$$

where $\bar{\tau}_i$ is the mean ion relaxation time, determined by expression (12.10).

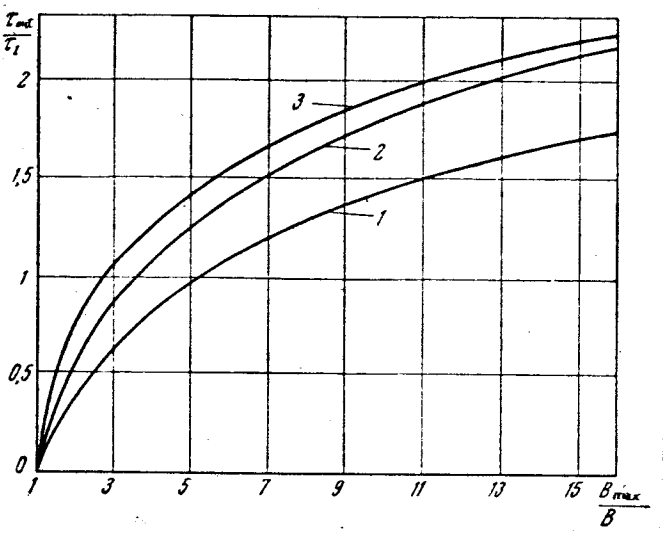

Figure 11.

In a similar way, for injection which is perpendicular to the magnetic field, we find the following from formula (18.33)

$$\tau_{out} = 1{,}57 \cdot \frac{\ln \frac{1}{\sin \vartheta_0}}{\cos \vartheta_0} \bar{\tau}_i. \qquad (18.36)$$

The graphs of these functions are shown in Figure 11 [curve *1* corresponds to formula (18.35); curve *2* - (18.37); curve *3* - (18.36)], while the *corkscrew ratio* $\frac{B_{max}}{B}$, which is connected with the limiting angle ϑ_0 by the relationship (18.1), is plotted on the abscissa axis.

10. In connection with the expressions derived for N and τ_{out}, it should be noted that they can be approximately correct only for *small angles of* ϑ_0, i.e., for *large values of the corkscrew ratio* $\frac{B_{max}}{B}$. In fact, if the angle ϑ_0 is very small, then the distribution function $F(\vartheta, p)$, as can be seen from formulas (18.10), (18.26) and (18.32), will barely differ from an

isotropic distribution - with the exception of the region within the limiting cone and its close surroundings. But, as can be readily shown, the integrands in formulas (15.5) and (15.8) are finite. Therefore, the contribution which is made by this region and its surroundings to the tensor $D_{\alpha\beta}$ and the vector A_α, is small. Consequently, the real values of $D_{\alpha\beta}$ and A_α will barely differ from the Maxwell expressions which were used in the derivation. /169

In addition, in deriving all of the expressions for N and τ_{out}, we used the Landau equation, and, consequently, disregarded the effect of close collisions. However, it can be readily seen that for small angles of ϑ_0 the contribution which is made by the close collisions in the stream N is negligible with respect to that which is made by the far-removed collisions. In fact, for close collisions the particles are scattered at large angles. Therefore, the stream of particles - which pass through the limiting cone - which is caused by close collisions is approximately proportional to a solid angle at the apex of this cone $\Omega = 2\pi(1 - \cos \vartheta_0) = 4\pi \sin^2 \frac{\vartheta_0}{2} \approx \pi \vartheta_0^2$ - i.e., the square of the angle ϑ_0. If the angle ϑ_0 is small, then this stream becomes negligible as compared with the stream of particles which are leaving as a result of far-removed collisions. Actually, for small values of ϑ_0, the latter stream - as follows from expressions (18.22) and (18.33) - is proportional to $\frac{1}{\ln \frac{1}{\vartheta_0}}$. The ratio of the first stream to the second stream is proportional to $\vartheta^2 \ln \frac{1}{\vartheta_0}$, and strives to zero for $\vartheta_0 \to 0$.

Thus, formulas (18.35) and (18.36) can be assumed to be valid only for small angles of ϑ_0 - i.e., for large values of the corkscrew ratio $\frac{B_{max}}{B}$. For small ϑ_0, it is possible to assume $\cos \vartheta_0 = 1$, after which formula (18.36) changes to

$$\tau_{out} = 0{,}785\bar{\tau}_i \ln \frac{B_{max}}{B}, \qquad (18.37)$$

formula (18.35) under these same assumptions changes into

$$\tau_{out} = 0{,}785\bar{\tau}_i \ln \frac{B_{max}}{B} - \frac{1}{2}\bar{\tau}_i,$$

which barely differs from formula (18.37).

The graph for the function of (18.37) is shown in Figure 11 by curve *2*. It provides a correct asymptotic expression for the confinement time at large values of the corkscrew ratio $\frac{B_{max}}{B}$. However, utilizing this formula, we also obtain the correct value for τ_{out} in another limiting case, when $\frac{B_{max}}{B} = 1$. Actually, it is clear from physical considerations that in this limiting case the trap will absolutely not confine the particles - i.e. $\tau_{out} = 0$. But we obtain the same value for τ_{out} if we assume $\frac{B_{max}}{B} = 1$ in formula (18.37). Therefore, formula (18.37) can be used not only for large, but also for any, values of the corkscrew ratio, if we consider this formula as a *reasonable extrapolation.* /170

11. The diffusion mechanism for the particle outflow within the limiting cone thus leads to a very slight (approximately logarithmic) dependence of the confinement time τ_{out} on the corkscrew ratio $\frac{B_{max}}{B}$. *The confinement time is barely sensitive to the changes in the corkscrew ratio.* For this reason, it is not advisable to construct a trap with very large values of the corkscrew ratio. This fact obviously makes a trap with magnetic mirrors unfeasible as a thermonuclear reactor of the future, even if it were possible to suppress the different type of instabilities arising in such a trap.

In fact, in order that the trap function as a thermonuclear reactor, it is necessary that the confinement time τ_{out} *be not less than* the mean time τ_p which is required before an ion reacts with another ion

$$\tau_{out} > \tau_p. \tag{18.38}$$

For the time τ_p, it is possible to write

$$\tau_p = \frac{1}{n\langle\sigma u\rangle}, \tag{18.39}$$

where $\langle\sigma u\rangle$ is the product, which is averaged by the appropriate method, of the *reaction cross-section* σ and *relative velocity* u, which is only a function of *temperature* for any reaction with a Maxwell velocity distribution. As regards the time τ_{out}, according to formula (18.37) it can be identified with the mean ion relaxation time $\bar{\tau}_i$ (the corkscrew ratio $\frac{B_{max}}{B} \approx 3.5$ corresponds to this assumption). Under this assumption, we obtain the following from the formulas (12.10), (18.38) and (18.39) for the reaction *dd*

$$T_i > 2{,}7 \cdot 10^{-17} \left[\frac{L}{\langle\sigma u\rangle}\right]^{2/3} \tag{18.40}$$

(all quantities in the cgs system). The values of the quantity $\langle \sigma u \rangle$ for different temperatures with a Maxwell velocity distribution are given in the work of Tompson (Ref. 22). The maximum temperature in his calculations amounts to 10^9 °K. At this temperature, for the reactions dd we have $\langle \sigma u \rangle = 5 \cdot 10^{-17} \text{cm}^3 \text{sec}^{-1}$. If this value is used and we assume that $L = 20$, then from inequality (18.40) we obtain

$$T_i > 1{,}46 \cdot 10^{-5} \text{ erg} = 9{,}15 \text{ Mev} = 1{,}06 \cdot 10^{11} \text{°K}.$$

The fact that the temperature T_i was obtained above the original temperature (10^9 °K) points to the fact that we clearly used the wrong value for $\langle \sigma u \rangle$. However, this can not be of great importance, in the first place, because the quantity $\langle \sigma u \rangle$ increases slowly with a further increase in the temperature. In the second place, with a non-Maxwell velocity distribution of the reacting ions (and this will be the case in a thermonuclear reactor) the quantity $\langle \sigma u \rangle$ is less than for a Maxwell distribution (Ref. 18). For the reaction td the quantity $\langle \sigma u \rangle$ is one order of magnitude larger. Again, this cannot be of great importance, since in a reactor which is operating with a mixture of tritium and deuterium, the outflow of ions from the trap is caused by collisions between any plasma ions, while nuclear reactions only occur for collisions of tritium ions with deuterium ions. /171

We should note that in the reaction dd the charged reaction products - p, t, and He^3 - have kinetic energies which equal 3, 1, and 0.81 Mev, respectively.

In the reaction

$$d + t \rightarrow \text{He}^4 + n + 17{,}6 \text{ Mev}$$

α-particles are obtained with an energy of about 3.5 Mev. Thus, we arrive at the following conclusion. *In order that an adiabatic trap with magnetic mirrors serve as a thermonuclear reactor with a positive energy outflow , it is necessary to heat the plasma which is located in it up to the temperature at which the mean kinetic energy of each ion is not less than the energy (per one charged particle) which is liberated in the given thermal reactions.* It is difficult to see how, under these conditions, it is possible to heat the plasma, due to the nuclear processes which are taking place in it - i.e., to carry out a *self-sustaining thermonuclear reaction.*

19. THE NATURE AND ELIMINATION OF DIVERGENCE IN THE THEORY OF PAIR COLLISIONS

1. All of the results in the preceding sections were obtained with the *pair collision* approximation. This approximation, as we have seen, leads to *divergent integrals*, and the divergence is eliminated by *artificial truncation* of the radius of action of the Coulomb forces. Let us now turn in greater detail to the physical assumptions underlying the theory of pair collisions, and to the nature of the indicated divergence.

One of the basic assumptions in the theory of pair collisions is the assumption regarding the *instantaneous quality* of the collision act. As will be shown below, this assumption leads to divergence. Therefore, in order to understand the nature of the divergence and to determine the way to eliminate it, it is necessary to renounce this assumption and to take the fact into consideration that the interaction of a pair of particles is a *long* process and not an instantaneous process. This will be done in the present section. /172

As was shown in Section 14, the problem in this theory is reduced to calculating the mean values $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$, which are called *moments of the distribution function* of the first and second orders, respectively. Here, Δp is the impulse change of the test particle, which it undergoes during the small time τ as a result of *random* interactions with the other particles. The word "random" indicates that in the calculation of Δp the changes in the impulse p *must not be taken into consideration.* These changes are caused by the action of *regular* force fields. The effect of the latter is taken into consideration by the term $\frac{\partial}{\partial p_\alpha}(X_\alpha F)$ of the kinetic equation (14.9).

In calculating Δp, we confine ourselves to classical mechanics. It will be seen from the subsequent considerations, if the results of Section 5 are taken into account, that the consideration of the quantum properties of the particles appears only in the numerical values of the Coulomb logarithm, which appears in the theory.

2. Let us conceive of a frame of reference in which the plasma as a whole is at rest. We shall use E^i to designate the electric field created by the i^{th} particle at the point where the test particle is located. The total field, which is created by all the field particles at this point will equal $\sum_i E^i$, where the summation is made over all of the field particles. If a regular component is

present in a plasma of macroscopic charges, and if this component is contained in the sum $\sum_i E^i$ and equals the mean value of the indicated sum, it can differ from zero. However, without a loss of generality it can be assumed that this regular component equals zero, i.e.,

$$\left\langle \sum_i \mathbf{E}^i \right\rangle = 0. \tag{19.1}$$

In fact, it is a component of the *self-consistent field* and, as was already indicated, it is automatically taken into account by the term $\frac{\partial}{\partial p_\alpha}(X_\alpha F)$ of the kinetic equation (14.9). Therefore, it is necessary to take into consideration only the non-regular electric field which enters into the sum $\sum E^i$ - i.e., the field whose mean value equals zero. This field produces the random changes in Δp of the test particle impulse with which the values of the moments, which are interesting to us, $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$ are determined.

Let us introduce the hypothesis that the electric fields created by the field particles at the point where the test particle is located are *statistically independent*, or more precisely

$$\left\langle \sum_{i \neq j} \sum E^i_\alpha(t)\, E^j_\beta(t') \right\rangle = 0, \tag{19.2}$$

whatever the moments of time t and t' may be, as well as the coordinate indexes α and β. In terms of its nature, relationship (19.2) is an analog of the more general *hypothesis of molecular chaos* which is widely used in the *kinetic theory of gases*. /173

3. Let us now examine the motion of a test particle over a small period of time τ (from the moment of time t until the moment $t + \tau$) during which its trajectory is not significantly distorted. We shall assume that the initial direction of motion (at the moment t) passes along the z-axis. Then, due to the slight extent to which the trajectory is curved, each direction which is perpendicular to the z-axis can be assumed to be approximately perpendicular to the trajectory of the test particle at any point. The random change in the test particle impulse during the time τ is

$$\Delta p_\alpha = \sum_i \delta p^i_\alpha,$$

where

$$\delta p^i_\alpha = e \int_0^\tau E^i_\alpha(t + t')\, dt'.$$

Thus, we have

$$\Delta p_\alpha \Delta p_\beta = \int_0^\tau \int_0^\tau dt'\, dt'' \sum_i \sum_j E^i_\alpha(t + t')\, E^j_\beta(t + t'').$$

Averaging these expressions and utilizing the relationship (19.2), we obtain

$$\langle \Delta p_\alpha \rangle = \sum_i \langle \delta p_\alpha^i \rangle, \tag{19.3}$$

$$\langle \Delta p_\alpha \Delta p_\beta \rangle = \sum_i \sum_j \langle \delta p_\alpha^i \delta p_\beta^j \rangle. \tag{19.4}$$

Our scheme correctly presents the nature of the interaction between a test particle and the other plasma particles: the interactions overlap in time, and do not follow behind each other, as in the pair collision approximation. In other words, the interactions are *multiple*, and not pair interactions. However, the relationships (19.3) and (19.4) show that the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$ - when a test particle is scattered by a group of all field particles - are found by adding the moments obtained from scattering by individual field particles.

4. Nevertheless, this result is of little help to us in solving the problem of calculating the given moments, and it would be incorrect to assume that it reduces this problem to the problem of the interaction between two particles. Actually, in order to calculate $\delta^i p$ it is necessary to know beforehand the *true trajectories and velocities* of both the test particles and the i^{th} field particles. In order to do this, it would be necessary to solve the *N-body problem* - the problem of the motion of all the plasma particles which interact with each other and with the external fields. This has not been accomplished. A new probability assumption is necessary, which we shall introduce.

For this purpose, we shall divide the interactions between a test particle and each of the field particles into *close* and *far-removed* interactions. Although this division is conditional and contains a significant amount of uncertainty, this uncertainty does not influence the final result within the limits of the calculational accuracy which we have assumed.

For close interactions, the particles approach each other within small distances, and as a result of this there are significant changes in their impulses during the time under consideration τ. Therefore, during the interaction between a test particle and one of the field particles the influence of the other field particles, and the interaction itself, can be regarded as an *instantaneous collision*.

For far-removed interactions, the particles are consistently located far from each other, and the changes in their impulses during

the time τ - as a result of the interaction - are small. The trajectories of the interacting particles are slightly distorted, but their form is not significantly affected by the result of the interaction. The true trajectories of the interacting particles can be approximated by straight lines, or by any other lines which are slightly distorted. Therefore, the far-removed interactions can be considered in the *approximation of the given trajectories* of the interacting particles. In particular, such trajectories can be represented by the trajectories which would be obtained if the system were composed of only two bodies - of the test particle and the field particle under consideration. In Coulomb interactions, these trajectories will be hyperbolic.

Thus, in a calculation of the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$, both the close and the far-removed interactions can be ultimately reduced to *independent interactions* of a pair of particles - i.e., to those interactions, when each pair of particles is regarded as an isolated system during the interaction time. We have now arrived at the *approximation of pair interactions* [1].

5. However, there is a significant difference between this *pair interaction approximation* and the *pair collision approximation*, which we utilized in the preceding sections. In the pair collision approximation, the real interaction between two particles is replaced by the instantaneous collision act, and, in addition, the impulse change $\delta^i p$ of the test particle, resulting from the collision, is assumed to equal the change which results in reality during the infinite interaction time. In the more general approximation of pair interactions, such a substitution is not made - in a calculation of the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$ the *actual* changes $\delta^i p$ in the impulse, during the finite interaction time for a pair of particles τ, must be used. The same is true for /175

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[1] It must be kept in mind that the method for calculating the random interactions with the aid of vector $\mathbf{I}$ in the kinetic equation (14.9) is suitable only for far-removed interactions, which are accompanied by small changes in the impulses of the interacting particles. But, with the Coulomb forces, the effect of close interactions is, as a rule, small as compared with the effect of the far-removed interactions. Therefore, the close interactions can be entirely disregarded, or they can be replaced by the equivalent far-removed interactions. This can be done by taking both the far-removed interactions and the close interactions into consideration in a calculation of the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$. This method, which is used in the present article, does not pretend to provide a fully adequate description of the operation.

all field particles.

For close interactions, both methods lead to almost the same values of $\delta^i p$. In this case, the time τ can be regarded as infinitely large, only if the moment of maximum convergence for the interacting particles occursat the mean time interval τ , or is close to it.

The picture is entirely different for far-removed interactions. Whatever the time interval τ may be, it is always possible to find a field particle which is far enough away, so that - when scattering is effected by it - the impulse change in the test particle during the time τ will be negligible, as compared with its change during an infinite interaction time. In this case, the first method of investigation (which is used in the approximation of pair collisions), based on the Rutherford formula, is not used. It utilizes exaggerated values for $\delta^i p$, which leads to divergent expressions for the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$, and for the quantities related to them. The divergence is not caused by the slowness with which the Coulomb forces decrease with distance, as is frequently stated, but is caused by the *incorrectness of the calculation* , in which the Rutherford formula - or expressions similar to it - are used *outside the limits of their applicability*. In a correct calculation, no divergences arise. Let us use the example of the calculation of the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$ to illustrate this.

6. Let us begin by calculating the change $\delta p_\perp$ in the perpendicular component of the test particle impulse, caused by its interaction with the field particle during the time from $t = 0$ to $t = \tau$. Thus we are employing a frame of reference in which the field particle at the moment of time $t = 0$ is at rest, and is located at the origin at this moment. The direction of the test particle velocity at this moment of time is assumed to be along the positive direction of the z-axis.

For our purposes, it is sufficient to confine ourselves to the far-removed interactions. Therefore, the calculation can be carried out in the approximation of the given test particle trajectory m . Let us use a straight line, which is parallel to the test particle velocity at the moment of time $t = 0$ and which is removed from the z-axis by the *aiming distance* ρ (Figure 12), as such a trajectory. Let the initial value of the z-coordinate of the test particle equal z_0 . Then, in the approximation of the given trajectory, its coordinate z at any moment of time t is determined by the expression $z = z_0 + ut$. An increase in $p_\perp$ during the time dt

$$dp_\perp = \frac{ee^*}{r^2}\sin\varphi\,dt = \frac{ee^*}{r^3}\varrho\,dt = \frac{ee^*\varrho}{(\varrho^2+z^2)^{3/2}}\cdot\frac{dz}{u},$$

and a change in $p_\perp$ during the time from $t = 0$ to $t = \tau$, give /176

$$\delta p_\perp = \frac{ee^*\varrho}{u}\int_{z_0}^{z_0+u\tau}\frac{dz}{(\varrho^2+z^2)^{3/2}}.$$

Carrying out the integration, we find

$$\delta p_\perp = \frac{ee^*}{u\varrho}\left\{\frac{z_0+u\tau}{\sqrt{\varrho^2+(z_0+u\tau)^2}} - \frac{z_0}{\sqrt{\varrho^2+z_0^2}}\right\}. \qquad (19.5)$$

If we set $z_0 = -\infty$, $z_0 + u\tau = +\infty$, in this formula, it changes to

$$\delta p_\perp = \frac{2ee^*}{u\varrho}.$$

The latter expression is used in the approximation of pair collisions.

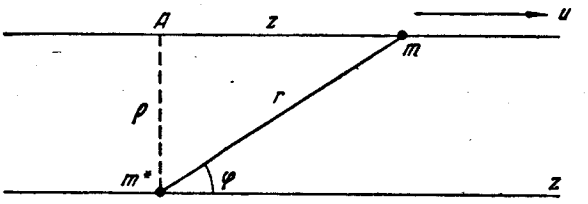

Figure 12.

7. Let us now calculate the moments of the second order $\langle \Delta p_\alpha \Delta p_\beta \rangle$. Let us first investigate the special case when all the field particles are identical and move at velocities which are identical in terms of magnitude and direction. Then the calculations can be most readily effected within the frame of reference where they are at rest. This we shall do. The initial direction of the relative velocity u of a test particle is assumed to lie along the z-axis of a rectangular coordinate system. Out of all of the second-order moments, in view of the symmetry, only $\langle \Delta p_x^2 \rangle$, $\langle \Delta p_y^2 \rangle$ and $\langle \Delta p_z^2 \rangle$ can differ from zero. In view of the same symmetry, they are related by the relationship

$$\langle \Delta p_x^2 \rangle = \langle \Delta p_y^2 \rangle = \frac{1}{2}\langle \Delta p_\perp^2 \rangle. \qquad (19.6)$$

Therefore, the calculations of $\langle \Delta p_x^2 \rangle$ and $\langle \Delta p_y^2 \rangle$ are reduced to a calculation of $\langle \Delta p_\perp^2 \rangle$.

The quantity $\Delta p_\perp$ can be represented in the form $\Delta p_\perp = \Delta_1 p_\perp + \Delta_2 p_\perp$, where $\Delta_1 p_\perp$ is the change in the perpendicular

impulse component of the test particle, which is caused by far-removed interactions, and $\Delta_2 p_\perp$ - is that caused by close interactions. In view of formula (19.4), we have

$$\langle \Delta p_\perp^2 \rangle = \langle \Delta_1 p_\perp^2 \rangle + \langle \Delta_2 p_\perp^2 \rangle. \tag{19.7}$$

8. First, let us calculate $\langle \Delta_1 p_\perp^2 \rangle$. We shall designate the far-removed interactions as interactions having aiming distances ρ which exceed a rather large length $d >> \rho_\perp$, where $\rho_\perp$ is given by expression (5.2). All other interactions will be called close interactions. The distance d must be large enough so that the calculation of $\langle \Delta_1 p_\perp^2 \rangle$ can be carried out in the given trajectory approximation of the test particle, which we shall assume to be rectilinear. The magnitude of d does not have to be defined more precisely, since it does not enter into the final results. /177

On the basis of formula (19.4), we have

$$\langle \Delta_1 p_\perp^2 \rangle = \sum_{\rho > d} \langle \delta^i p_\perp^2 \rangle, \tag{19.8}$$

where the summation is made over all the field particles, for which the aiming distance ρ exceeds d. The expression for $\delta^i p_\perp$ is given by formula (19.5). Let us approximate the sum (19.8) by the integral. The mean number of field particles in the volume element $dV = 2\pi\rho d\rho dz_0$ equals $n^* dV$. Therefore, the sum (19.8) changes into

$$\langle \Delta_1 p_\perp^2 \rangle = \frac{2\pi n^* (ee^*)^2}{u^2} \int_d^\infty \frac{d\rho}{\rho} \int_{-\infty}^{+\infty} \left[\frac{z_0 + u\tau}{\sqrt{\rho^2 + (z_0 + u\tau)^2}} - \frac{z_0}{\sqrt{\rho^2 + z_0^2}} \right]^2 dz_0. \tag{19.9}$$

with approximation by the integral.

Let us introduce new integration variables x and α with respect to the formula

$$z_0 = \rho x, \quad \rho = \frac{u\tau}{\alpha}. \tag{19.10}$$

Then, we have

$$\langle \Delta_1 p_\perp^2 \rangle = \frac{2\pi n^* (ee^*)^2}{u} \tau \int_0^{\frac{u\tau}{d}} J(\alpha) \frac{d\alpha}{\alpha^2}, \tag{19.11}$$

where

$$J(\alpha) = \int_{-\infty}^{+\infty} \left[\frac{x + \alpha}{\sqrt{1 + (x + \alpha)^2}} - \frac{x}{\sqrt{1 + x^2}} \right]^2 dx. \tag{19.12}$$

The last integral is reduced to an *elliptic* integral. It can be readily calculated for small and large values of the parameter α. For this purpose, we shall introduce the notation

$$\varphi(x) = \frac{x}{\sqrt{1 + x^2}}.$$

Then the integrand in formula (19.12) can be written in the form $[\varphi(x+\alpha)-\varphi(x)]^2$. For small α, it can be approximated by the expression $\left[\frac{d\varphi}{dx}\alpha\right]^2 = \frac{\alpha^2}{(1+x^2)^3}$. For such α,

$$J(\alpha) = \alpha^2 \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{8}\alpha^2. \tag{19.13}$$

Let us now obtain the *asymptotic expression* for $J(\alpha)$ for large α. For this purpose, let us turn to the drawing (Figure 13).

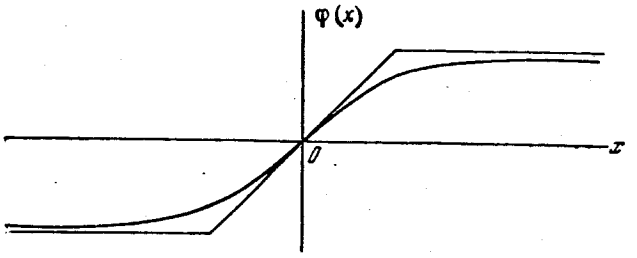

Figure 13.

The function $\varphi(x)$ asymptotically strives to $+1$ for $x \to +\infty$, and to -1 for $x \to -\infty$. For small x it can be approximated by the *linear function* $\varphi(x) = x$. If both of these approximations are extrapolated to the region of intermediate values of x, then we obtain the broken line shown in Figure 13. We shall replace this broken line by the curve $\varphi(x) = \frac{x}{\sqrt{1+x^2}}$ in calculating the integral (19.12). This substitution is barely expressed in the asymptotic value of expression (19.12) for large values of the parameter α. In order to illustrate this, Figure 14 shows the function $\varphi(x+\alpha)-\varphi(x)$ for $\alpha = 7$. The function approximating it is shown by the broken line. In a large part of the interval $-(\alpha+1) < x < 0$, which represents the main contribution to the integral (19.12) - the approximating function, which is larger than the true function, barely differs from the latter (the difference is noticeable only close to the edges of the indicated interval). This leads to a small exaggeration of the integral (19.12). Outside of the interval $-(\alpha+1) < x < 0$, where the integrand in formula (19.12) is small - the approximating function is smaller than the true function. This leads to a small underestimation of the integral (19.12). As a result, for large α the given approximation leads to an asymptotic expression for the integral $J(\alpha)$ whose accuracy is completely adequate for our purposes. In addition, as is clear from the following calculations, the approximation of the function $\varphi(x+\alpha)-\varphi(x)$ by the broken line, which is shown in Figure 13,

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cannot change the order of magnitude of $J(\alpha)$, either for small or intermediate values of α. Therefore, in order to have a single simple analytical expression for the function $J(\alpha)$, we shall use the indicated approximation *for any*, and not only for small, values of the parameter α. The error arising from this is unimportant, because the integral (19.11) will interest us for large values of the upper limit $\frac{u\tau}{d}$.

Thus, we approximate $\varphi(x)$ by the function

$$\varphi(x)=\begin{cases}-1 & \text{for } -\infty<x\leqslant -1,\\ x & \text{for } -1\leqslant x\leqslant +1,\\ +1 & \text{for } +1\leqslant x\leqslant +\infty.\end{cases}$$

Then we can readily obtain

$$J(\alpha)=\begin{cases}2\alpha^2-\frac{1}{3}\alpha^3 & \text{for } \alpha\leqslant 2,\\ 4\alpha-\frac{8}{3} & \text{for } \alpha\geqslant 2,\end{cases} \tag{19.14}$$

$$\langle\Delta_1 p_\perp^2\rangle=$$

$$=\begin{cases}\frac{8\pi n^*(ee^*)^2}{u}\tau\left[\frac{u\tau}{2d}-\frac{1}{24}\left(\frac{u\tau}{d}\right)^2\right] & \text{for } \frac{u\tau}{d}\leqslant 2,\\ \frac{8\pi n^*(ee^*)^2}{u}\tau\left[\ln\frac{u\tau}{d}+\frac{1}{2}-\ln 2+\frac{2}{3}\frac{d}{u\tau}\right] & \text{for } \frac{u\tau}{d}\geqslant 2^1.\end{cases} \tag{19.15}$$

The linear term 4α enters into the asymptotic formula (19.14) with the correct coefficient 4. Therefore, the logarithmic term $\tau\ln\frac{u\tau}{d}$ in formula (19.15) also has the correct coefficient. All of the other terms in formulas (19.14) and (19.15), are, strictly speaking, incorrect. They correctly impart only the order of the corresponding magnitudes. Consequently, only the second formula (19.15) /180

[1] The expression (19.14) for small α strives to zero in the same way as α^2, in conformance with the correct formula (19.13). However, instead of the correct coefficient $\frac{3\pi}{8}$, the coefficient 2 is obtained. This divergence can be eliminated, if the more general approximation of the function $\varphi(x)$ is used:

$$\varphi(x)=\begin{cases}-1 & \text{for } -\infty<x<-m,\\ \frac{x}{m} & \text{for } -m\leqslant x\leqslant +m,\\ +1 & \text{for } +m\leqslant x<+\infty,\end{cases}$$

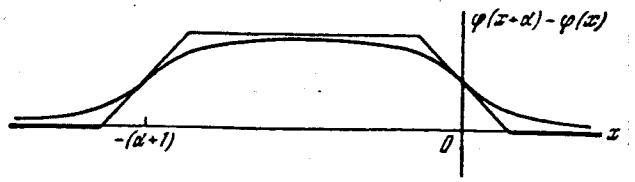

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Figure 14.

can be regarded as correct for large values of the time τ, when /180
$\ln \frac{u\tau}{d} >> 1$. It is just this case in which we are most interested. As will be seen further on, we are primarily interested in the case

(Footnote continued from previous page)

where m is the constant. Then, instead of expressions (19.14) and /179
(19.15), we obtain

$$J(\alpha) = \begin{cases} \frac{2}{m}\alpha^2 - \frac{1}{3m^2}\alpha^3 & \text{for } \alpha \leqslant 2m, \\ 4\alpha - \frac{8}{3}m & \text{for } \alpha \geqslant 2m, \end{cases} \tag{19.14a}$$

$$\langle \Delta_1 p_\perp^2 \rangle = \begin{cases} \frac{8\pi n^* (ee^*)^2}{u}\tau\left[\frac{u\tau}{2md} - \frac{1}{24}\left(\frac{u\tau}{md}\right)^2\right] & \text{for } \frac{u\tau}{d} \leqslant 2m, \\ \frac{8\pi n^* (ee^*)^2}{u}\tau\left[\ln\frac{u\tau}{d} + \frac{1}{2} - \ln(2m) + \frac{2m}{3}\frac{d}{u\tau}\right] & \text{for } \frac{u\tau}{d} \geqslant 2m. \end{cases} \tag{19.15a}$$

If we set $m = \frac{16}{3\pi}$, then for $\alpha \to 0$ formula (19.14a) changes into the correct formula (19.13). In this way, we obtain the correct expressions for $\langle \Delta_1 p_\perp^2 \rangle$ for small and large values of the parameter $\frac{u\tau}{d}$. For intermediate values of this parameter, formulas (19.15a) can be regarded as extrapolation formulas.

when the time τ, which is contained in the logarithmic multiplier $\ln \frac{u\tau}{d}$, is on the order of the period of *Langmuir fluctuations* for electrons, and consequently, $u\tau$ is on the order of the *Debye radius*. Therefore, assuming that $\ln \frac{u\tau}{d} >> 1$ and keeping the fact in mind that the auxiliary term $\left(\frac{1}{2} - \ln 2 + \frac{2}{3}\frac{d}{u\tau}\right)$ is small and is not calculated absolutely correctly, we can discard this small term and can write

$$\langle \Delta_1 p_\perp^2 \rangle = \frac{8\pi n^* (ee^*)^2}{u} \tau \ln \frac{u\tau}{d}. \tag{19.16}$$

9. Let us now calculate $\langle \Delta_2 p_\perp^2 \rangle$, under the same assumption that $\frac{u\tau}{d} >> 1$. Under this assumption, the time τ can be assumed to be almost infinitely large, and we can calculate the mean value $\langle \Delta_2 p_\perp^2 \rangle$ in which we are interested, utilizing the concept of the effective scattering cross-section $\sigma(\vartheta\ u)$, which is determined by Rutherford's formula (3.5). In view of formula (2.4), the change in the perpendicular component of the test particle impulse resulting from a single scattering act is determined by the expression

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$$\delta p_\perp = \mu \delta u_\perp = \mu u \sin \vartheta, \tag{19.17}$$

where ϑ is the scattering angle. As is known from the elementary theory of Coulomb scattering, it is connected with the aiming distance ρ by the relationship

$$\operatorname{tg} \frac{\vartheta}{2} = \frac{\varrho_\perp}{\varrho}. \tag{19.18}$$

Thus, we have

$$\langle \Delta_2 p_\perp^2 \rangle = n^* u \tau \int \delta p_\perp^2 \cdot \sigma(\vartheta,\ u)\, d\Omega,$$

and the integration must be extended to the region of the scattering angles $\vartheta_0 \leqslant \vartheta \leqslant \pi$. Here, ϑ_0 is the scattering angle value which it assumes for $\rho = d$. After substituting expression (3.5) in the preceding formula, we obtain

$$\langle \Delta_2 p_\perp^2 \rangle = \frac{4\pi n^* (ee^*)^2}{u} \tau \int_{\vartheta_0}^{\pi} \frac{\cos^3 \frac{\vartheta}{2}}{\sin \frac{\vartheta}{2}} d\vartheta =$$

$$= \frac{8\pi n^* (ee^*)^2}{u} \tau \left\{ -\ln \sin \frac{\vartheta_0}{2} - \frac{1}{2} \cos^2 \frac{\vartheta_0}{2} \right\}$$

or

$$\langle \Delta_2 p_\perp^2 \rangle = \frac{8\pi n^* (ee^*)^2}{u} \tau \left\{ \ln \frac{\sqrt{d^2 + \varrho_\perp^2}}{\varrho_\perp} - \frac{d^2}{2(d^2 + \varrho_\perp^2)} \right\}. \tag{19.19}$$

Since $d >> \rho_\perp$, we can disregard all the terms, with the exception of the logarithm. This is permissible, especially since the interactions which are accompanied by large impulse changes (on the order of the impulse itself) are all taken into account in an equally incorrect manner by the theory, since the kinetic equation (14.4) is derived under the assumption that the impulse changes are small during particle interactions. Thus, under the condition $u\tau >> d$ we have

$$\langle \Delta_2 p_\perp^2 \rangle = \frac{8\pi n^* (ee^*)^2}{u} \tau \ln \frac{d}{\rho_\perp}. \tag{19.20}$$

Now combining expressions (19.16) and (19.20), we obtain the following in view of relationship (19.7):

$$\langle \Delta p_\perp^2 \rangle = \frac{8\pi n^* (ee^*)^2}{u} \tau \ln \frac{u\tau}{\rho_\perp}. \tag{19.21}$$

The indefinite, auxiliary quantity d is eliminated from the final result, in accordance with the indications given above. Formula (19.21) was first obtained by V. I. Kogan (Ref. 23) by a different method. We obtain the following from this, in view of the relationship (19.6): /182

$$\langle \Delta p_x^2 \rangle = \langle \Delta p_y^2 \rangle = \frac{4\pi n^* (ee^*)^2}{u} \tau \ln \frac{u\tau}{\rho_\perp}. \tag{19.22}$$

10. Let us now calculate the moments of the first order Δp_α and also the moment $\langle \Delta p_z^2 \rangle$. In view of the symmetry of the coordinate system which we have chosen, out of all the first-order moments only $\langle \Delta p_z \rangle$ can differ from zero. Since

$$\mathbf{u} = \mathbf{v} - \mathbf{v}^* = \frac{\mathbf{p}}{m} - \frac{\mathbf{p}^*}{m^*},$$

and the change $\delta \mathbf{p} + \delta \mathbf{p}^*$ in the total impulse during pair collisions equals zero, then

$$\delta \mathbf{p} = -\delta \mathbf{p}^* = \mu \delta \mathbf{u}. \tag{19.23}$$

Let us turn to Figure 12. The increases in the relative velocity in the sections of the given rectilinear trajectory which lie to the right and to the left of point A, have opposite signs. It thus follows that the increases δp_z, which are caused by different field particles, can be positive or negative with equal probability. Therefore, it can be shown that the moment $\langle \Delta p_z \rangle = \langle \sum^* \delta p_z \rangle$ becomes zero. This is not the case in actuality. In this discussion, attention has been given to the change in the vector $\mathbf{u}$ in terms of magnitude, but its change in terms of direction has not been taken into consideration. This discussion only proves that the *change in the length of the vector* $\mathbf{u}$ *does not play a role in the calculation of the moment* $\langle \Delta p_z \rangle$. This change in the length of the vector $\mathbf{u}$ can

be disregarded; it is only necessary to take into account the *changes in the vector* $\mathbf{u}$ *with respect to direction.* Doing this, we can write

$$\mathbf{u}^2 = (\mathbf{u} + \delta\mathbf{u})^2$$

or

$$2(\mathbf{u}\cdot\delta\mathbf{u}) = -(\delta\mathbf{u})^2.$$

Thus, in view of relationship (19.23), we have

$$2(\mathbf{u}\cdot\delta\mathbf{p}) = -\frac{1}{\mu}(\delta\mathbf{p})^2.$$

Finally,

$$\delta p_z = -\frac{(\delta\mathbf{p})^2}{2u\mu} \tag{19.24}$$

in accordance with formula (2.16). It follows from formula (19.24), in the first place, that the quantity δp_z is of a much higher order of smallness than $\delta\mathbf{p}$. Consequently, Δp_z is of a higher order of smallness as compared with $\Delta p_\perp$. Therefore, in the approximation which we are using it can be assumed that

$$\langle \Delta p_z^2 \rangle = 0. \tag{19.25}$$

In the second place, making a summation of expression (19.24) with respect to all the field particles, we obtain /183

$$\langle \Delta p_z \rangle = -\frac{1}{2u\mu}\langle \Delta p_\perp^2 \rangle = -\frac{4\pi n^* (ee^*)^2}{\mu u^2}\tau \ln\frac{u\tau}{\varrho_\perp}. \tag{19.26}$$

11. Now, we no longer have to employ the special coordinate system in which the calculations were carried out, and we can write the expressions for the vector $\langle \Delta p_\alpha \rangle$ and the tensor $\langle \Delta p_\alpha \Delta p_\beta \rangle$ in an arbitrary, rectangular coordinate system. This can be accomplished in the same way as in Section 15. As a result, we obtain

$$\langle \Delta p_\alpha \rangle = -\frac{4\pi n^* (ee^*)^2}{\mu u^3} u_\alpha \tau \ln\frac{u\tau}{\varrho_\perp}, \tag{19.27}$$

$$\langle \Delta p_\alpha \Delta p_\beta \rangle = 4\pi (ee^*)^2 u_{\alpha\beta}\tau \ln\frac{u\tau}{\varrho_\perp}, \tag{19.28}$$

where the tensor $u_{\alpha\beta}$ is given by the previous expression (15.4).

We must now remove the last restriction - the assumption concerning the uniformity of all the field particles and their identical velocities. For this purpose, it is sufficient to replace n^* by $F^*(\mathbf{p}^*)\,d\mathbf{p}^*$ in formulas (19.27) and (19.28), to integrate with respect to $\mathbf{p}^*$, and then to make a summation with respect to all the types of field particles. This gives

$$\langle \Delta p_\alpha \rangle = -\tau \sum\nolimits^{*} 4\pi (ee^*)^2 \int \frac{u_\alpha}{\mu u^3} \ln \frac{u\tau}{\varrho_\perp} F^* (\mathbf{p}^*)\, d\mathbf{p}^*, \tag{19.29}$$

$$\langle \Delta p_\alpha \Delta p_\beta \rangle = \tau \sum\nolimits^{*} 4\pi (ee^*)^2 \int u_{\alpha\beta} \ln \frac{u\tau}{\varrho_\perp} F^* (\mathbf{p}^*)\, d\mathbf{p}^*. \tag{19.30}$$

12. Thus, we have obtained *final*, and not divergent, expressions for the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$. In spite of this, we must still overcome the difficulty of the theory of pair interactions. Actually, if we substitute expressions (19.29) and (19.30) in formulas (14.11) and (14.12), expressions which *clearly contain the time interval* τ are obtained for the dynamic friction coefficient A_α and the diffusion tensor $D_{\alpha\beta}$. This interval can be selected in any way - it must only be not too small and not too large. Therefore, the *indefinite quantity* τ - which can have *any values whatever*, which is physically meaningless - is contained in expression (14.10) for the current density in impulse space I_α and in the kinetic equation (14.10).

Within the framework of the pair interaction approximation, two methods for overcoming this difficulty can be pointed out.

13. In the first place, it is possible to *artificially truncate* the radius of action of the Coulomb forces at a certain magnitude of D. Then the total moment $\langle \Delta p_\perp^2 \rangle$ can be accurately calculated, in the same way as formula (19.20) was derived. In practical terms, this means that the interactions are replaced by instantaneous collisions.

The distance D is contained in the integrands (19.29) and (19.30), /184 instead of the quantity $u\tau$, and no results which are physically meaningless are obtained. This course was selected in the theory of pair collisions, which we discussed in the preceding sections.

14. In the second place, it is possible to manage without this artificial truncation, and to proceed in the following way. *Expressions (19.29) and (19.30) cannot be valid for arbitrarily large values of the time* τ. This can be seen from the fact that for large τ the test particle trajectory deviates from its original direction by significant angles, and for this reason the calculation in the given trajectory approximation is not applicable. In actuality, due to the *collective effects* in the plasma, formulas (19.29) and (19.30) become inapplicable much sooner. Be that as it may, there is a finite time τ_0 (we shall call it the *collision time*) at which, for $\tau > \tau_0$, formulas (19.29) and (19.30) are no longer valid, while for $\tau = \tau_0$ they can be assumed to be applicable. Assuming this, we can reason as follows.

Let the time interval τ_0 be contained in τ a whole number (k) of times. We can write it in the form $\tau = \tau_1 + \tau_2 + \cdots + \tau_k$, while each of the intervals τ_i equals τ_0. We shall use $\Delta^i p_\alpha$ to designate the change in the test particle impulse during the time interval τ_i. Then, we have

$$\Delta p_\alpha = \sum_{i=1}^{k} \Delta^i p_\alpha,$$

$$\Delta p_\alpha \Delta p_\beta = \sum_{i=1}^{k} \sum_{j=1}^{k} \Delta^i p_\alpha \Delta^j p_\beta.$$

If the collision time τ_0 is sufficiently large, the changes $\Delta^i p_\alpha$ and $\Delta^j p_\beta$ in the test particle impulse during two different time intervals τ_i and τ_j are *statistically independent:*

$$\langle \Delta^i p_\alpha \Delta^j p_\beta \rangle = 0.$$

There, averaging the preceding expressions, we obtain

$$\langle \Delta p_\alpha \rangle = \sum_{i=1}^{k} \langle \Delta^i p_\alpha \rangle = k \langle \Delta^i p_\alpha \rangle = \frac{\tau}{\tau_0} \langle \Delta^i p_\alpha \rangle,$$

$$\langle \Delta p_\alpha \Delta p_\beta \rangle = \sum_{i=1}^{k} \langle \Delta^i p_\alpha \Delta^i p_\beta \rangle = k \langle \Delta^i p_\alpha \Delta^i p_\beta \rangle = \frac{\tau}{\tau_0} \langle \Delta^i p_\alpha \Delta^i p_\beta \rangle.$$

We have taken the fact into account here that the moments $\langle \Delta^i p_\alpha \rangle$ and $\langle \Delta^i p_\alpha \Delta^i p_\beta \rangle$ are one and the same for all i. The values of these moments can be found according to formulas (19.29) and (19.30), if τ is replaced by τ_0 in them.

As a result, we arrive at the expressions /185

$$\langle \Delta p_\alpha \rangle = -\tau \sum\nolimits^{*} 4\pi (ee^*)^2 \int \frac{u_\alpha}{\mu u^3} \ln \frac{u\tau_0}{\rho_\perp} F^*(\mathbf{p}^*)\, d\mathbf{p}^*, \tag{19.31}$$

$$\langle \Delta p_\alpha \Delta p_\beta \rangle = \tau \sum\nolimits^{*} 4\pi (ee^*)^2 \int u_{\alpha\beta} \ln \frac{u\tau_0}{\rho_\perp} F^*(\mathbf{p}^*)\, d\mathbf{p}^*. \tag{19.32}$$

These expressions are *not sensitive* to the collision times τ_0. Thus, one of the conditions for their applicability is that the logarithmic multiplier $\ln \frac{u\tau_0}{\rho_\perp}$ *is large*. Therefore, the assumption which was used in the derivation regarding the briefness of the times τ and τ_0 is unimportant and unnecessary; formulas (19.31) and (19.32) are valid when this assumption is not fulfilled.

Formulas (19.31) and (19.32) lead to the same expressions (15.5) and (15.6) for the diffusion tensor and the dynamic friction coefficient, which were obtained previously by truncating the radius of action of the Coulomb forces. The following quantity plays the role of the Coulomb logarithm:

$$L = \ln \frac{u\tau_0}{\varrho_\perp}. \tag{19.33}$$

It can be replaced by a certain mean value and, just as was done previously, it can be removed from inside the integral sign.

15. It now only remains to determine the value of the collision time τ_0. This problem, as well as the rigorous substantiation of the collision time concept itself, can only be solved within the framework of the *N-body theory*. However, the time τ_0 can be roughly estimated without going into this theory in detail.

If the same displacement is transmitted to all the plasma electrons at a certain moment of time, then - as is well-known - an electric field arises in the plasma which is proportional to this displacement. Under the influence of this field, the plasma electrons begin to make harmonic fluctuations with the *Langmuir* or *plasma* frequency ω which is determined by the expression

$$\omega^2 = \frac{4\pi e^2 n}{m}. \tag{19.34}$$

Fluctuations with this frequency arise if, at a certain moment of time, small displacements in the *radial* direction are transmitted to all the plasma electrons; the magnitude of these displacements depends only on the distance r from the center O of the corresponding sphere. Let us assume that the plasma is electrically neutral at a certain moment of time. Let us conceive of a certain sphere S in it with the macroscopic radius r and with the center at the point O (Figure 15). The electric charge and the electric field within this sphere equal zero at the moment of time under consideration; let us now assume that all the electrons undergo *small displacements* in the direction of the sphere radii (it can be assumed that the ions are infinitely heavy and fixed). Then the electrons, /186
which were previously located on the surface of the sphere S, pass over to the sphere S', which is concentric with it and which has the radius $r' = r + x$. The electric charge within the sphere S decreases by the quantity $q = 4\pi r^2 ex$. As a result, the following electric field arises on the surface of the sphere S

$$E = \frac{q}{r^2} = 4\pi ex,$$

which is directed toward the center of the sphere. Under the influence of this field, the electrons begin to make *radial harmonic fluctuations* with a plasma frequency which is determined by formula (19.34).

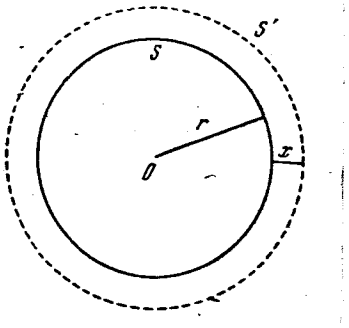

Figure 15.

This discussion shows that for definite types of collective motions - for example, *translational and radial* motions - the plasma electrons behave, not like free particles, but like *quasi elastically-connected* particles. But each field particle of the plasma tends to produce collective radial electron fluctuations with the center at the point where this particle is located. The *quasi elastic forces* which thus arise influence not only the electrons, but also the plasma ions. In particular, the test particle is subjected to their influence. The simplest method for calculating the collective effects in a plasma consists of taking these quasi elastic forces into account in an examination of the test particle motion. They were not taken into account when formulas (19.29) and (19.30) were derived. If the quasi elastic force equals zero at a certain moment of time, then after one fourth a period of Langmuir fluctuations $\frac{T}{4} = \frac{\pi}{2\omega}$ in terms of magnitude it equals the Coulomb force with which a test particle acts upon a field particle. In general, during the time $\frac{1}{\omega}$ the quasi elastic force changes by a magnitude on the order of the Coulomb force. Therefore, it can be expected that formulas (19.29) and (19.30) are valid for small time intervals $\tau \sim \frac{1}{\omega}$. On this basis, we can determine the collision time τ_0 by the formula

$$\tau_0 = \frac{1}{\omega}. \tag{19.35}$$

It is not important to define the numerical coefficient which is contained in formula (19.35) more precisely, since the time τ_0 enters into formulas (19.31) and (19.32) in the argument of the logarithm.

The *thermal motion* of a test particle and a field particle interacting with it has not been taken into consideration in this discussion. However, this is not of great importance, since we are interested in *far-removed interactions*, when the distance between interacting particles is large as compared with the distance $u\tau_0$ which is traversed by the test particle during the time τ_0 during its motion with respect to the field particle. /187

If the velocity u is now replaced by a certain mean electron velocity in formula (19.33) - for example, by its *most probable velocity* $\sqrt{\frac{2T}{m}}$ - then we obtain

$$u\tau_0 = \sqrt{\frac{T}{2\pi ne^2}} = 2D, \tag{19.36}$$

where D is the Debye radius. Therefore, the mean Coulomb logarithm, calculated with the aid of formula (19.33) barely differs from that which was obtained previously by truncation of the radius of action of the Coulomb forces by the *Debye length* D. Thus, it has been shown that both methods for eliminating the uncertainties in the expressions for the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$, and in all of the other formulas at the same time, lead to identical results.

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